A diagrammatic approach to Hecke algebras of the reflection equation

2000 J. Phys. A: Math. Gen. 33 1265

(http://iopscience.iop.org/0305-4470/33/6/312)

J. Phys. A: Math. Gen. **33** (2000) 1265–1296. Printed in the UK PII: S0305-4470(00)06705-6

A diagrammatic approach to Hecke algebras of the reflection equation

Paul P Martin, David Woodcock and Dan Levy

Mathematics Department, City University, Northampton Square, London ECIV 0HB, UK

Received 5 August 1999

Abstract. Hecke algebra deformations of the wreath product $C_d \wr S_n$ arise in solutions to the Yang–Baxter equations with boundary (the reflection equation). We use a simple diagrammatic approach to construct bases for these algebras. We hence introduce quotient algebras suitable for constructing physical solutions to the reflection equation, with a well-defined thermodynamic (large n) limit. For all d and n we determine the generic representation theory of these algebras, developing a formalism suitable for the analysis of corresponding q-spin-chain Hamiltonian and transfer matrix spectra.

1. Introduction

Let C_d denote the cyclic group of order d, and $C_d \wr S_n$ the *wreath product* with the symmetric group S_n. Just as the ordinary Hecke algebra H_n is a q-deformation of the group algebra of S_n [23], so a Hecke algebra of $C_d \wr S_n$ is a multiparameter deformation of the group algebra of $C_d \wr S_n$ [10]. These algebras are important for solvable models in two-dimensional statistical mechanics because, just as representations of H_n may be used to construct solutions to the Yang–Baxter equations (YBEs) [7]

$$R_i(\theta_1)R_{i+1}(\theta_1+\theta_2)R_i(\theta_2) = R_{i+1}(\theta_2)R_i(\theta_1+\theta_2)R_{i+1}(\theta_1) \tag{1}$$

$$[R_i(\theta_1), R_j(\theta_2)] = 0 \qquad i \neq j \pm 1 \tag{2}$$

so *their* representations may be used [29] to construct solutions to the YBE with boundary; i.e., with the reflection equation (RE) [11, 19, 42]:

$$R_1(\theta_1-\theta_2)K(\theta_1)R_1(\theta_1+\theta_2)K(\theta_2) = K(\theta_2)R_1(\theta_1+\theta_2)K(\theta_1)R_1(\theta_1-\theta_2) \tag{3}$$

$$[R_i(\theta'), K(\theta)] = 0 \qquad i \geqslant 2. \tag{4}$$

These algebras include Broué and Malle's *cyclotomic* Hecke algebras of type $G(d, 1, n)$ [10] here denoted G_n^d, and the specialized 'Hecke algebra extension' [29, theorem 1] (case $k = d-1$) here denoted D_n^d.

In this paper we consider the problem of constucting *physically useful* solutions to RE using G_n^d and D_n^d. The main difficulty here is similar to one which arises in solving the bare YBE using H_n [23, 32]. That is, there is no obvious thermodynamic or large-n limit algebra—cf for example, the well-behaved Temperley–Lieb algebra constructions, which may be tied directly to the Bethe ansatz [4, 28, 45]. As the Temperley–Lieb example suggests, a resolution of the problem for H_n lies in the *representation* obtained from $U_q sl_N$ invariant vertex models [34]. Fixing N, this representation is very far from faithful in general, but the corresponding quotient algebra, denoted H_n^N, is still richly interesting (indeed the Temperley–Lieb algebra is the case $N = 2$), and has a well-defined large-n limit. In this paper we generalize this well-behaved quotient to the cyclotomic algebras, and hence develop a direct analogue of the representation theory tools used in [34, 35] to analyse quantum spin chain Hamiltonian spectra. We do this, however, without needing to construct an analogue for the role of $U_q sl_N$.

(In addition to their role in solvable models it is also thought that these algebras may be useful in constructing knot invariants in higher genus [27], and in asymmetric diffusion modelling with generalized boundary conditions [2, 3]. For further examples of potential physics applications see [14, 15, 20, 30, 38–40, 42]. The cyclotomic algebra has also been the subject of some beautiful and rarefied mathematical work [5, 6, 16, 17].)

If $\{A_n | n = 1, 2, \ldots\}$ is a sequence of groups or algebras obeying $A_i \subseteq A_{i+1}$ we write A_- for this sequence. In [34, 35] it was shown how to construct a global ('thermodynamic') limit H_*^N for the sequence of Hecke algebra quotient algebras H_-^N. This result was used to determine quantum spin chain level crossings. In seeking the same facility for cyclotomic Hecke algebras G_n^d (resp. D_n^d), or indeed generally, there are a number of technical challenges to be overcome:

- A quotient algebra $\overline{\mathsf{G}}_n^d$ of G_n^d with a global (large n) limit must be constructed (cf [34]).
- In particular, globalization and localization functors must be constructed (relating the operators for measurement of given observables, such as spin–spin correlations, on *different* lattice sizes)—this means we need an idempotent $Y \in \overline{\mathsf{G}}_r^d$ (some $r > 0$) such that $Y\overline{\mathsf{G}}_{n+r}^d Y \cong \overline{\mathsf{G}}_n^d$ for all n [34, 35].
- The representation theory of the quotient $\overline{\mathsf{G}}_n^d / \overline{\mathsf{G}}_n^d Y \overline{\mathsf{G}}_n^d$ must be determined (giving the iterative step in the representation theory analysis of $\overline{\mathsf{G}}_n^d$ by iteration on n).
- Ideally, we must construct a vertex model (tensor space) representation (cf [41]); and a weight lattice indexing simple modules, with a geometry induced by the limit form of induction and restriction of generically irreducible representations in the sequence $\overline{\mathsf{G}}_n^d (n = 1, 2, \ldots)$.

This paper is in two parts. In sections 2 and 3 we develop the diagrammatic basis of D_n^d which is to be our main computational tool. First we give a diagrammatic representation of a spanning set—*braid diagrams* [9] are already familiar in the physics literature [26], and we use variations on these. We then determine linear independence by reference to the 'generic' representation theory of D_n^d, a 'deformation' of the standard representation theory of $\mathbb{C}C_d \wr S_n$ (cf [6]).

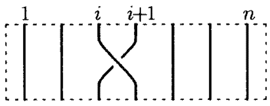
In the second part (sections 4 and 5) the challenges above are dealt with for the generic algebras, i.e. for q not a root of unity and all other parameters also generic (by using diagrams we develop a formalism suitable for generalizing to the exceptional cases, but these will be dealt with specifically elsewhere). The main technical results are encapsulated in propositions 19 and 21. A striking feature is the extra layer of symmetry in the $\overline{\mathsf{D}}_n^d$ formalism cf H_n^N, the potential utility of which is exemplified in an application to H_n^N in section 5.

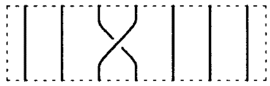
The remainder of this section is occupied with a review of essential background material and the motivation and definition of D_n^d.

1.1. Braid diagrams and the Hecke algebras H_n

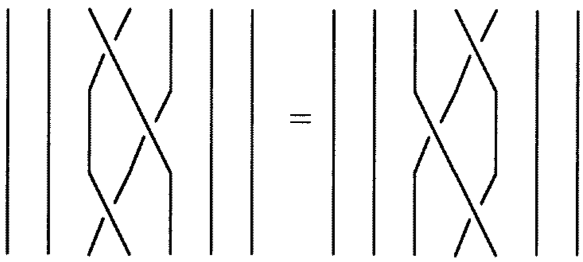
We use the notations of [34] unless otherwise stated. For K a ring and G a set, KG denotes the free K-module with basis G.

The n-string *braid group* $\mathcal{B}_n$ [9, 26] may be generated by a set of adjacent pair braiding operators $\{g_i, g_i^{-1} | i = 1, 2, \ldots, n-1\}$ represented by diagrams of the form:

$g_i =$ 

$g_i^{-1} =$  (5)

(hereafter we will omit the bounding box). Composition is by vertical juxtaposition (consider $g_i g_i^{-1} = 1$), and braid equivalences are illustrated, for example, by the *braid relation*:

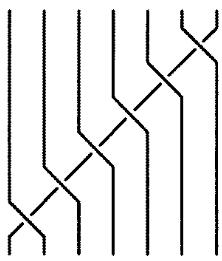
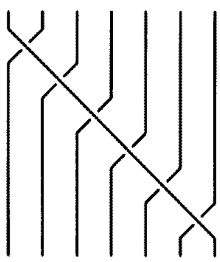


(6)

The strings in a braid may be labelled 1 to n as indicated in (5). A *pure braid* is an element of $\mathcal{B}_n$ in which the order of the string labels as read across the bottom of the braid is again 1 to n (for example, g_i^2 is a pure braid). It will be evident that the set of pure braids is a normal subgroup of $\mathcal{B}_n$. The quotient group is S_n [9].

Let A be a K-algebra defined by generators and relations. For $w \in A$ a word in the generators write w^T for the word obtained by writing these generators in the reverse order. Suppose that the set of relations is invariant under T (as in $\mathbb{C}\mathcal{B}_n$, for example). For $x \in A$ let x^T denote the K-linear extension of this operation. Then if $M \subseteq A$ is a left A-module, M^T is a right A-module.

Define $G_1 = 1$ and $G_m = g_{m-1} G_{m-1}$, i.e.

$G_n =$  $G_n^T =$ 

and define

$$W_m = \{1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_{m-1}\} = \{G_1^T, G_2^T, G_3^T, \ldots, G_m^T\}.$$

The inner automorphism $\Lambda : b \mapsto G_n^{-1} b G_n$ of $\mathcal{B}_n$ takes the subgroup $\mathcal{B}_m \subset \mathcal{B}_n$ braiding the first m strings ($m < n$) to an isomorphic subgroup braiding strings 2 to $m + 1$. We write $b \mapsto b^{(1)}$ for the image of $b \in \mathcal{B}_m$, $b^{(2)}$ for $(b^{(1)})^{(1)}$ (where defined) and so on.

For q indeterminate let $\mathbb{C}_q$ denote the ring of Laurent polynomials $\mathbb{C}[q, q^{-1}]$. The *Hecke algebra* H_n is the quotient of group algebra $\mathbb{C}_q \mathcal{B}_n$ by the ideal generated by element $(g_1 + q^2)(g_1 - 1)$ [8]. This H_n is a free $\mathbb{C}_q$-module of rank $n!$ [23], in which the generators obey relations

$$(g_i + q^2)(g_i - 1) = 0. \tag{7}$$

Some workers use $t_i = qg_i$ obeying $(t_i + q)(t_i - q^{-1}) = 0$; and it is often convenient to work instead with $U_i = q^{-1}(1 - g_i)$ obeying

$$U_i^2 = (q + q^{-1})U_i. \tag{8}$$

Let K be a field such that there is a ring homomorphism

$$f : \mathbb{C}[q, q^{-1}] \to K.$$

Then K is a $\mathbb{C}[q, q^{-1}]$-module by restriction, and we can define K-algebra ${}^K H_n = K \otimes_{\mathbb{C}[q,q^{-1}]} H_n$. In particular, for $K \supset \mathbb{C}[q, q^{-1}]$ put ${}^K H_n =: H_n(q)$, and for $K = \mathbb{C}$ and $f : q \mapsto q_c \in \mathbb{C} \setminus \{0\}$ put ${}^K H_n =: H_n(q_c)$. For example, $H_n(1) \stackrel{\sim}{=} \mathbb{C}S_n$.

The set of braids generated using the g_i pictures in equations (5), (6) span H_n (and of course any other $\mathcal{B}_n$ quotient algebra). Although *not* in general linearly independent in the quotient these braid diagrams still form a useful pictorial realization of H_n, and it is possible to identify subsets which *are* linearly independent (some of which subsets are still spanning). For example, consider the group homomorphism from the braid group onto the symmetric group

$$\begin{aligned} P &: \mathcal{B}_n \to S_n \\ P &: b \mapsto P(b) \end{aligned} \tag{9}$$

that is, where the permutation $P(b)$ takes i to j if the string in braid b starting at position i at the top runs to position j at the bottom.

Proposition 1. *Any subset T of the domain of P for which the map is a set injection is still linearly independent in the quotient H_n.*

To see this recall $\mathbb{C}S_n \stackrel{\sim}{=} H_n(1)$. Suppose there is a linear dependence in the image of T in H_n—then there is still a linear dependence with $q = 1$†.

For R a ring and $a, b, \ldots, c \in R$ define $\Psi(a, b, \ldots, c) = ab \ldots c$. A natural basis of H_n [23] is the *basis of reduced words* $B_n^{\text{red}} = \Psi(\times_{m=1}^{n} W_m^{(n-m)})$. For example, $\Psi(W_1^{(1)} \times W_2^{(0)}) = \{1, g_1\}$. There is a length function $\text{len}(b)$ on this basis given by the number of factors g_i (so $\text{len}(1) = 0$).

1.2. Motivation: Solutions to the YBEs

Put $q = \exp(\mathrm{i}\gamma)$ and define $[x] = \frac{\sin x\gamma}{\sin\gamma}$ (to be understood as a limit if $\gamma = 0$), so $2\cos\gamma = q + q^{-1}$. Then putting $U_i = q^{-1}(1 - g_i)$ in $H_n(q)$ we find that

$$R_i(\theta) = \sin(\gamma + \theta)1 - \sin(\theta)U_i \tag{10}$$

gives a solution to equations (1) and (2). By varying the representation of $H_n(q)$ such solutions include the ones appearing in the critical Potts [7], Andrews–Baxter–Forrester [4], vertex [41] and other physical models. Putting equation (10) into equation (3) we get

$$\begin{aligned} &\sin(\theta_1 + \theta_2)\sin(\theta_1 - \theta_2)[U_1 K(\theta_1) U_1, K(\theta_2)] \\ &\quad = \sin(\gamma + \theta_1 - \theta_2)\sin(\theta_1 + \theta_2)(K(\theta_1)U_1 K(\theta_2) - K(\theta_2)U_1 K(\theta_1)U_1, K(\theta_1)) \\ &\qquad + \sin(\gamma + \theta_1 + \theta_2)\sin(\theta_1 - \theta_2)(U_1 K(\theta_1)K(\theta_2) - K(\theta_2)K(\theta_1)U_1) \\ &\qquad - \sin(\gamma + \theta_1 + \theta_2)\sin(\gamma + \theta_1 - \theta_2)[K(\theta_1), K(\theta_2)]. \end{aligned} \tag{11}$$

This is non-trivial to analyse, but generators and relations for extensions of $H_n(q)$ which give solutions for a variety of q values are observed in [29]. (Note that these solutions obey $[K(\theta_1), K(\theta_2)] = 0$.) Some examples now follow.

† Possibly a given linear relation may factor by $(q - 1)$—then one must divide through by $(q - 1)$ before putting $q = 1$, and so as a proof this is incomplete. We will omit discussion of the torsion-freeness of H_n as a $\mathbb{C}_q$-module in the interests of brevity—see [23, 36].

Definition 1. *Fix $x = (x_1, \ldots, x_d)$ a d-tuple of scalar parameters such that $[x_i - x_j] = 0$ implies $i = j$. Algebra $\mathsf{D}_n^d(q, x)$ has generators $\{1, U_1, \ldots, U_{n-1}, v(1), \ldots, v(d)\}$ and relations $U_i^2 = [2]U_i$, $U_iU_{i+1}U_i - U_i = U_{i+1}U_iU_{i+1} - U_{i+1}$, $[U_i, U_{i+j}] = 0$ $(j > 1)$, $\sum_i v(i) = 1$, $[U_i, v(j)] = 0$ $(i > 1)$,*

$$v(i)v(j) = \delta_{ij}v(j) \tag{12}$$

$$[U_1v(k)U_1, v(l)] = \frac{[x_k - x_l - 1]}{[x_k - x_l]}(v(k)U_1v(l) - v(l)U_1v(k)) \qquad (k \neq l).$$

(This is [29] theorem 1 case $\gamma \neq 0$ and $k = d - 1$—the variable c_{ij} there is given by $c_{ij} = -\mathrm{i}\sin\gamma\frac{q^m+q^{-m}}{q^m-q^{-m}}$ where $m = x_i - x_j$; and

$$K(\theta) = \sum_{i=1}^{d} w_i(\theta)v(i)$$

where the $w_i(\theta)$ are certain scalar functions. Note that corollary 1.1 of [29] is false except in this case.)

Definition 2. *For $\lambda = (\lambda_1, \ldots, \lambda_d)$ a d-tuple of scalar parameters the cyclotomic Hecke algebra $\mathsf{G}_n^d = \mathsf{G}_n^d(q, \lambda)$ is the extension of the generators and relations of $H_n(q)$ by a new generator X obeying $[g_i, X] = 0$ $(i > 1)$,*

$$\prod_{j=1}^{d}(X - \lambda_j 1) = 0 \tag{13}$$

$$[g_1Xg_1, X] = 0. \tag{14}$$

(This is also known as the Ariki–Koike algebra [6].) It will be evident that these definitions are self-consistent, and that both algebras contain $H_n(q)$ as a subalgebra. However, the result implicit in [29] that these algebras may be given the same basis for any choice of parameters is not obvious—indeed, the stronger implicit claim that a more general algebra has this property is false (see section 2.2). Identifying the copies of $H_n(q)$ in each algebra we may map between them when $\lambda_i = q^{2x_i}$ by $X \mapsto \lambda_i v(i)$ (see section 5.1).

Recall that S_n has generators $\sigma_i = P(g_i) = (i\ i{+}1)$ (this last is the *cycle notation* [21]).

Definition 3. *The group $C_d \wr S_n$ is the extension of S_n by a generator τ, with relations $\tau^d = 1$, $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$ (cf the reflection equation (3)) and $[\tau, \sigma_i] = 0$ $(i > 1)$.*

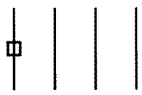
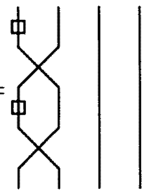
For example, $C_2 \wr S_n$ is the hyperoctahedral group [6, 10, 22].

Note, with $\lambda_j = \mathrm{e}^{2\pi\mathrm{i}j/d}$ and $q = 1$ (resp. $x_j = j/2d$ and, as it were, $q = \mathrm{e}^{2\pi\mathrm{i}}$) G_n^d (resp. D_n^d) maps isomorphically to the group algebra over $\mathbb{C}$ of $C_d \wr S_n$ [6, 10] (cf equation (13)). We will show in section 2.2 that $\mathsf{D}_n^d(q, x)$ may be given the same basis for each choice of the parameters ($\mathsf{G}_n^d(q, \lambda)$ has the same property [6]). These algebras are, in that sense, *deformations* of $\mathbb{C}C_d \wr S_n$, and a study of their representation theory is informed by a study of $C_d \wr S_n$.

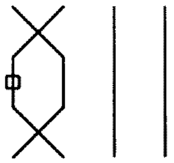
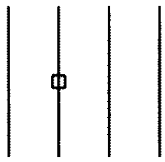
1.3. The conjugacy classes of $C_d \wr S_n$

The generators σ_i of S_n may be represented as in equation (5), but ignoring the 'over/under' information, since by equation (9) we have $P(g_i) = P(g_i^{-1})$. In these pictorial terms $\tau \in C_d \wr S_n$

may be thought of as a 'bead' living on the leftmost string. Thus we have (here with $n = 4$)

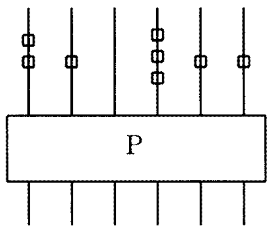
$\tau =$  $\tau\sigma_1\tau\sigma_1 =$ 

and so on. Put $\tau_1 = \tau$ and $\tau_i = \sigma_{i-1}\tau_{i-1}\sigma_{i-1}$ ('Murphy elements'). Depict $\tau_2 = \sigma_1\tau\sigma_1$ by

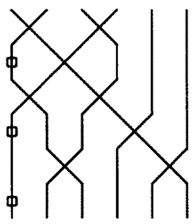
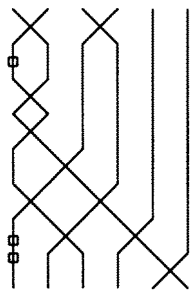
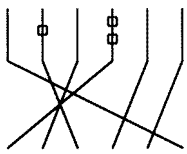
 $=$  (15)

Pursuing this realization, the bead can be *moved onto any different string* by conjugating by an appropriate permutation, and τ_i is a bead on the ith string. The consistency of this picture is ensured by the relation $[\sigma_1\tau\sigma_1, \tau] = 0$, which implies $[\tau_i, \tau_j] = 0$.

Clearly every element of $C_d \wr S_n$ can be arranged (e.g. here with $n = 6$) in the form

$w = \left(\prod_{i=1}^{n} \tau_i^{\alpha_i}\right) P =$  (16)

(where $P \in S_n$) by pushing all the beads on the strings to the top. For example,

 $=$  $= \ldots.. =$ 

where the intermediate step shown is the explicit verification, via equation (15), of the move used on the first bead. Then immediately

$$|C_d \wr S_n| = |S_n| d^n = n! d^n. \tag{17}$$

Recall that conjugacy classes of S_n may be characterized by the unique cycle structure of their elements [21]. Possible cycle structures are indexed by the Young diagrams of order n. Thus the equivalence classes of simple modules of S_n are indexed by the same set [13, 21].

Definition 4. *Let* $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^d)$ *be an ordered* d*-tuple of Young diagrams, with* $|\lambda| = \sum_i |\lambda^i| = n$*. Then* Γ_n^d *is the set of all such* d*-tuples.*

Suppose we represent $P \in S_n$ as a product of commuting cycles. If the element $w \in C_d \wr S_n$ is as in equation (16) then conjugation by $T \in S_n$ would not change the cycle structure, nor change the total number of beads attached to a given cycle (and so neither would conjugation by any element of $C_d \wr S_n \supset S_n$). Thus a conjugacy class of $C_d \wr S_n$ is characterized by the cycle structure together with a *colour* or *weight* from $1, 2, \ldots, d$ for each cycle. Sorting the cycles into sets of fixed weight we arrive at the following proposition.

Proposition 2. *Conjugacy classes of $C_d \wr S_n$ are in 1–1 correspondence with d-tuples of Young diagrams of summed degree n. Equivalence classes of simple modules, both for $C_d \wr S_n$ and for its group algebra over $\mathbb{C}$, are also indexed by Γ_n^d.*

2. On constructing deformations of $\mathbb{C}C_d \wr S_n$

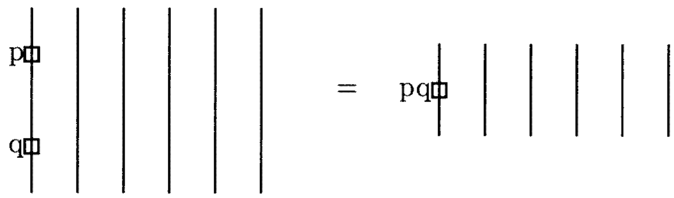
Consider deforming $\mathbb{C}C_d = \mathbb{C}C_d \wr S_1 \subseteq \mathbb{C}C_d \wr S_n$ as follows. Fix $d \in \mathbb{N}$ and ring K, and for $\alpha = (\alpha_1, \ldots, \alpha_d)$ indeterminates in K let $T[\alpha]$ be the K-algebra generated by $1, \tau_g$ obeying

$$\prod_{i=1}^{d} (\tau_g - \alpha_i 1) = 0. \tag{18}$$

This $T[\alpha]$ has basis $\{1, \tau_g, \ldots, \tau_g^{d-1}\}$. Let $\kappa_j = \prod_{i \neq j} (\alpha_j - \alpha_i)$. For $j = 1, \ldots, d$ the elements $v_u(j) = \prod_{i \neq j} (\tau_g - \alpha_i 1)$ obey $v_u(j) v_u(k) = \kappa_j \delta_{jk} v_u(j)$, and are distinct. That is, if κ_j is invertible in K then $\kappa_j^{-1} v_u(j)$ is a primitive central idempotent. If $K = \mathbb{C}$ we think of $T[\alpha]$ as providing a set of algebras, one for each point α in parameter space $\mathbb{C}^d$, each with the same basis. A *specialization* of $T[\alpha]$ then provides a closed subset of this set (usually, but not necessarily, the algebra at a single point in parameter space). A specialization $T[\alpha^s]$ of $T[\alpha]$ over $\mathbb{C}$ is semisimple provided that $\alpha_i^s = \alpha_j^s$ only if $i = j$. Since this condition is satisfied on a (Zariski) open subset of parameter space it is the *generic* situation. Conversely, the $\{v_u(j)\}$ are no longer all distinct in the non-semisimple specializations $\alpha_i^c = \alpha_{i+1}^c$, some i. Either way, when $K = \mathbb{C}$ and $\alpha_i = \lambda_i$, $T[\alpha] \cong \mathsf{G}_1^d \hookrightarrow \mathsf{G}_n^d$.

On the other hand, let $V = \mathsf{D}_1^d$, the $\mathbb{C}$-algebra generated by $v(1), \ldots, v(d)$. If A is a commutative, semisimple, unital $\mathbb{C}$-algebra of dimension d it is isomorphic to V. In particular, the *generic* specializations of $T[\alpha]$, such as $\mathbb{C}C_d$, are isomorphic to V.

Let A be a d-dimensional unital algebra over $\mathbb{C}$. We may generalize the group algebras of the sequence $C_d \wr S_- = \{C_d \wr S_1 \subset \cdots \subset C_d \wr S_n\}$ (note $C_d \cong C_d \wr S_1$) to a sequence of $\mathbb{C}$-algebras $\mathcal{H}_-^A = \{\mathcal{H}_1^A \subset \cdots \subset \mathcal{H}_n^A\}$ ($A \cong \mathcal{H}_1^A$) as follows. Start with S_n and introduce, for each $p \in A$, an element $\tau_p \in \mathcal{H}_n^A$ such that $p \mapsto \tau_p$ is an algebra homomorphism of A into $\mathcal{H}_n^A$ (e.g. there is a relation $\tau_p \tau_q = \tau_{pq}$, depicted:



cf section 1.3) and

$$\tau_{p_1} \sigma_1 \tau_{p_2} \sigma_1 = \sigma_1 \tau_{p_2} \sigma_1 \tau_{p_1} \tag{19}$$

(the reader may care to draw the diagram for this relation).

Proposition 3. *Let τ_g and 1 generate A. Then $\tau_g \sigma_1 \tau_g \sigma_1 = \sigma_1 \tau_g \sigma_1 \tau_g$ implies equation (19).*

Henceforward let A also be commutative. Then A could be any $T[\alpha^s]$. In this case there is a parameter space of algebras $\mathcal{H}_n^A$ including $\mathbb{C}C_d \wr S_n$, and they are collectively a deformation of $\mathbb{C}C_d \wr S_n$.

We call a sequence of algebras D_- a *Hecke-type* deformation of $\mathcal{H}_-^A$, cf Hoefsmit [22], if: (I) a set of generators of D_n are the generators of H_n plus the generators of A; (II) A and H_n are subalgebras of D_n; (III) the identity elements of D_n, A and H_n coincide; (IV) $[g_i, A] = 0$ for $i > 1$; and (V) $D_{n-1} \subset D_n$ (note that $D_1 = A$).

To generate such a deformation we can proceed by writing down certain new relations involving g_1 and A (defining D_2), and then checking for consistency.

2.1. The case $n = 2$

For K a field and D any K-algebra which is a quotient by some relations $\sim$ of the free product of two subalgebras $\langle H, A\rangle$ (say), with D, H, A all having the same unit, then trivially there is an inclusion of K-modules:

$$\langle H, A\rangle/\sim \qquad \to HAHAHAHA\ldots. \tag{20}$$

Proposition 4. *Let $D(v)$ be a K-algebra which is a quotient of the free product $\langle H_2, A\rangle$, and in which as K-modules*

$$AH_2AH_2 \leqslant H_2AH_2A. \tag{21}$$

Then $D(v)$ has dimension $\leqslant 2d^2$.

Proof. Any relations such that equation (21) is satisfied allow us to truncate the set of words required to span the rhs of equation (20) at H_2AH_2A. Since $\dim(H_2) = 2$ and $\dim(A) = d$ we require no more than $2d^2$ words. □

We now construct relations $\sim$ such that $D(q = 1) = \mathcal{H}_2^A$. One way to proceed is to identify A with $T[\alpha]$ and require equation (14) (see [6]). However, it turns out that A can take an important *physical* role, so here instead we take A to be commutative and semisimple. We thus have $A \cong \mathbb{C}C_d \cong V$. The image of $\{v_u(1), v_u(2), \ldots, v_u(d)\}$ under this isomorphism is an *orthogonal* (and normalizable) basis for A. (If A is non-semisimple such a basis does not exist—for example, consider $\alpha_1 = \alpha_2$. Thus the algebra we build *cannot* be identical to the Ariki–Koike algebra, but merely generically isomorphic to it.) Without further loss of generality consider the relations on H_2 and A separately in the form $U_1^2 = [2]U_1$ and $v(i)v(j) = \delta_{ij}v(i)$. There are at most $2d$ inequivalent one-dimensional representations of *any* algebra $D(v)$ obeying these, given by $i = 1, 2, \ldots, d$ in

$$R_{+i}(U_1) = 0 \qquad R_{-i}(U_1) = [2] \tag{22}$$

$$R_{\pm i}(v(j)) = \delta_{ij} \tag{23}$$

(new relations might kill some of these representations—such relations would not be consistent with specialization to $\mathcal{H}_2^A$, since *this* clearly has $2d$ one-dimensional representations: i.e., precisely those obtained by putting $q = 1$ in the above†). Further, it is easy to verify that any two-dimensional irreducible representation of such an algebra can be written in the form

$$R_{ij}^a(U_1) = \begin{pmatrix} a & a([2]-a) \\ 1 & [2]-a \end{pmatrix} \tag{24}$$

$$R_{ij}^a(v(k)) = \begin{pmatrix} \delta_{ik} & 0 \\ 0 & \delta_{jk} \end{pmatrix} \tag{25}$$

† These $\pm i$ indices correspond to $\lambda = (0, 0, \ldots, 0, (2), 0, \ldots, 0)$ and $\lambda = (0, 0, \ldots, 0, (1^2), 0, \ldots, 0)$, respectively, in the classification scheme for irreducibles to be given in section 3.

where $i \neq j$ and a is some scalar. In fact, for an isomorphism with $\mathcal{H}_2^A$ we require $d(d-1)/2$ such representations, and these may without loss of generality be indexed by all ordered pairs (i, j) such that $d \geqslant j > i \geqslant 1$ (again, cf section 3†). We can see this as follows: putting $[2] = 2$, $a = 1$ each of these is readily checked (by taking the trace of $v(i)$) to be a distinct irreducible representation of $\mathcal{H}_n^A$. Taken with the one-dimensional representations this set saturates the total dimension of the algebra, i.e. $2d \cdot 1^2 + \binom{d}{2} \cdot 2^2 = 2d^2$, so there are no other irreducibles. In the deformation we are *ab initio* free to choose a distinct in each case (call it $a = a_{ij}$). Not withstanding this freedom, all the 'representations' $R_{\pm i}$ and $R_{ij}^{a_{ij}}$ obey

$$[U_1 v(k) U_1, v(l)] = \begin{cases} a_{kl} L_{kl} & (k < l) \\ ([2] - a_{lk}) L_{kl} & (l < k) \end{cases} \tag{26}$$

where

$$L_{ij} = v(i) U_1 v(j) - v(j) U_1 v(i)$$

by direct computation. That is, these relations are necessary for a deformation of Hecke type. On the other hand, they ensure equation (21), so equations (26), (8) and (12) are also *sufficient* to define the most general deformation of $\mathcal{H}_2^A$ of Hecke type (i.e. up to (IV)) in which A is semisimple. Let us call it $\mathsf{D}_2^d(q, a_-)$, with a_- symbolizing the list of indeterminate scalars a_{ij}.

The algebra $\mathsf{D}_2^d(q, a_-)$ has $\frac{d(d-1)}{2} + 1$ parameters (counting q). The algebras is section 1.2 have $d + 1$ parameters. The extra parameters will disappear once we apply condition (IV).

In terms of g_1 the relations (26) are

$$[g_1 v(k) g_1, v(l)] = \begin{cases} (1 - q a_{kl})(v(k) g_1 v(l) - v(l) g_1 v(k)) & (k < l) \\ (-q^2 + q a_{lk})(v(k) g_1 v(l) - v(l) g_1 v(k)) & (l < k) \end{cases} \tag{27}$$

or, equivalently,

$$[g_1 v(k) g_1, v(l)] = b_{kl}(v(k) g_1 v(l) - v(l) g_1 v(k)) \tag{28}$$

where $b_{kl} = (1 - q a_{kl})(k < l)$, and $b_{kl} = (-q^2 + q a_{lk})(l < k)$, so that $b_{kl} + b_{lk} = 1 - q^2$.

2.2. D_n^d diagrams, the case $n = 3$, and general n

Let D_-^d be a sequence of algebras such that $\mathsf{D}_2^d = \mathsf{D}_2^d(q, a_-^s)$, a_-^s some restriction of a_-.

Definition 5. *For $B \subseteq \mathsf{D}_{n-1}^d$ define $B_n^d(B) \subseteq \mathsf{D}_n^d$ by*

$$B_n^d(B) = \{v(i) G_n^T b, g_1 v(i) G_n^T b, g_2 g_1 v(i) G_n^T b, \ldots, G_n v(i) G_n^T b \mid b \in B, i = 1, 2, \ldots, d\} \tag{29}$$

and define $B_n^d = B_n^d(B_{n-1}^d)$ where $B_0^d = \{1\}$. (Note that $|B_n^d| = n! d^n$.)

The construction is illustrated schematically in the individual components of figure 1, where we introduce diagrams for elements and subsets of D_n^d. As for braid diagrams these diagrams are to be understood as being embedded in the plane, with implicit rectangular bounding box. In this scheme (cf the scheme in [34]) a *box* across the first m lines may represent an element or set of elements of D_m^d. A box across any other m adjacent lines may only represent elements of the corresponding H_m subalgebra. Note that a simply connected component of a diagram which component contains only string (no beads) may be manipulated as if it is a similar connected component of a braid diagram without changing the algebra element represented.

† The ij indices correspond to the positions of the non-zero entries in $\lambda = (0, 0, 0, \ldots, 0, (1), 0, \ldots, 0, (1), 0, \ldots, 0)$.

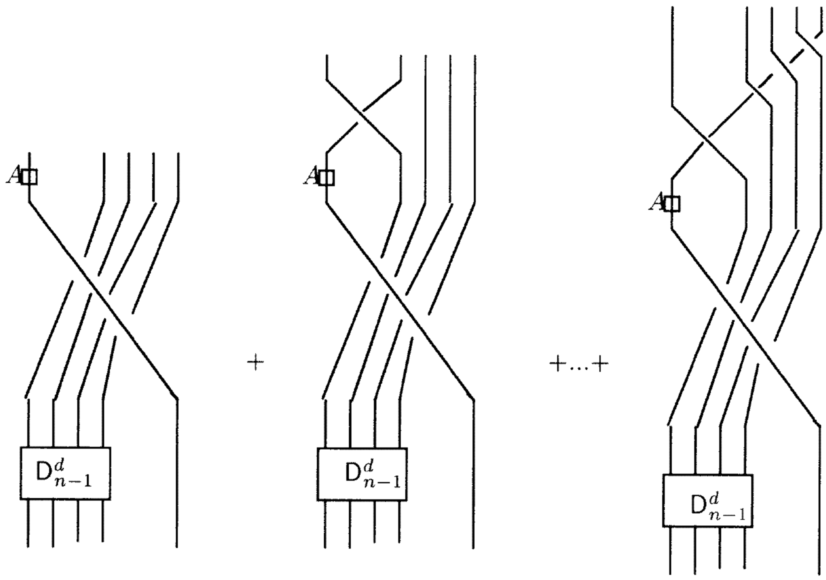


Figure 1. Pictorial representation of D^d_n decomposed as $\mathbb{C}$-submodules.

Proposition 5. *If B is a spanning set for D^d_{n-1} then $B^d_n(B)$ is a spanning set for D^d_n.*

Proof. The module spanned here may be depicted as in figure 1. We must show that this module is closed under, say, left (or, in the diagram, top) action of D^d_n. Now it is closed under action of $H_n \subset \mathsf{D}^d_n$ since

$$g_j(g_k g_{k-1}\dots g_1 A G^T_n \mathsf{D}^d_{n-1}) = (g_k g_{k-1}\dots g_1 A G^T_n \mathsf{D}^d_{n-1}) \qquad (j \neq k, k+1)$$

by $g_i \mathsf{D}^d_{n-1} = \mathsf{D}^d_{n-1} (i < n)$, and

$$g_k(g_k g_{k-1}\dots g_1 A G^T_n \mathsf{D}^d_{n-1}) \overset{\text{eqn (7)}}{\subseteq} (g_k g_{k-1}\dots g_1 A G^T_n \mathsf{D}^d_{n-1}) + (g_{k-1}\dots g_1 A G^T_n \mathsf{D}^d_{n-1}).$$

It closes under $v(i) \in A$ since by equation (26) or (27)

$$\begin{aligned} v(i)(g_1 A g_1 g_2 \dots g_{n-1}\mathsf{D}^d_{n-1}) &\overset{\text{eqn (26)}}{\subseteq} \\ &g_1 A g_1 A g_2 g_3 \dots g_{n-1}\mathsf{D}^d_{n-1} + A g_1 A g_2 g_3 \dots g_{n-1}\mathsf{D}^d_{n-1} \\ &\subseteq g_1 A g_1 g_2 g_3 \dots g_{n-1}\mathsf{D}^d_{n-1} + A g_1 g_2 g_3 \dots g_{n-1}\mathsf{D}^d_{n-1}. \end{aligned}$$

□

It follows that B^d_n spans D^d_n (the case $n = 1$ is clear).

Proposition 6. *The elements b of B^d_n may each be expressed in the form $b = h_b v_b$ where h_b is an element of the reduced word basis of H_n and v_b is a word of the form*

$$v_b = v(i_1 i_2 \dots i_n) := v(i_1)\underbrace{g_1 g_2 \dots g_{n-1}}_{G^T_n} v(i_2)\underbrace{g_1 g_2 \dots g_{n-2}}_{G^T_{n-1}} v(i_3)\dots v(i_{n-1}) g_1 v(i_n).$$

This decomposition is shown schematically in figure 2(a). The word v_b may be characterized by the list $(i_1, \dots, i_n)$ (i.e. reading the diagram from top to bottom). This list is called the *signature* of b.

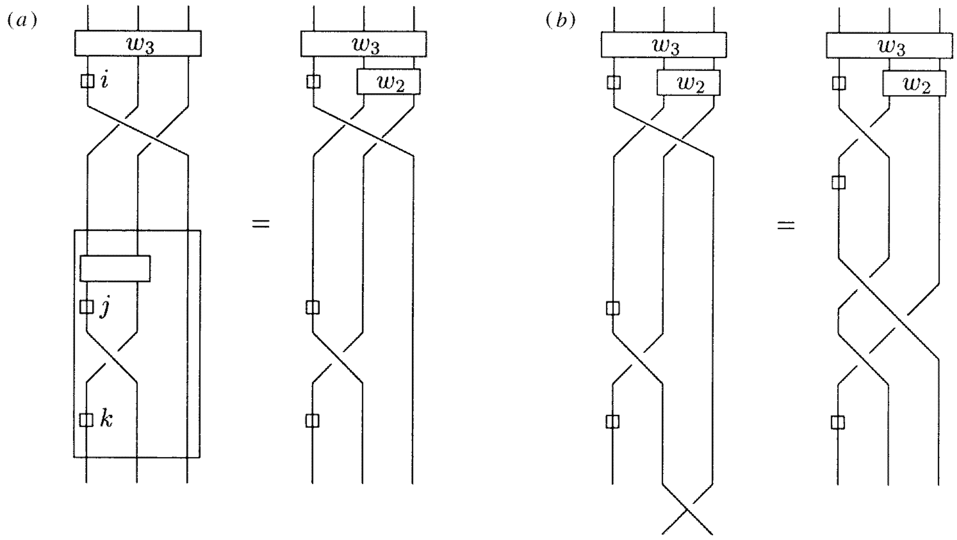


Figure 2. (a) Pictorial representation of a re-expression of B_3^d, exhibiting a basis of H_3 (elements of form $w_3 w_2^{(1)}$ with $w_m \in W_m^{(0)}$) as a 'factor'; and (b) pictorial representation of the (right) action of g_2 on B_3^d, exhibiting the reduction to a calculation equivalent to $g_1 B_2^d$.

Figure 2(b) shows schematically that the subset $B_{(n)}^d$ of elements of B_n^d for which the signature is a permutation of $\{1, 2, \ldots, n\}$ spans a right submodule, if it exists (i.e. if $d \geqslant n$). In particular, we see that the effect of g_2 acting from the right (from below) is to take $b = h_b v_b$ with signature (1,2,3) (say) to a linear combination of words with signatures (1,2,3) and (2,1,3). It follows from proposition 5 that B_n^d is linearly independent in $\mathbb{C}C_d \wr S_n$. We require it to be so in general. A necessary condition for this is that $B_{(n)}^d$ is a basis for a representation. An explicit construction shows that the right actions of the generators on this subset do *not* produce a representation of $\mathsf{D}_n^d(q, a_-)$. Specifically, $R(g_1g_2g_1 - g_2g_1g_2) = b_{123}M$ where $b_{123} = (b_{12}b_{23} + b_{21}b_{13} - b_{13}b_{23})$ and matrix $M \neq 0$. Thus we must require that the parameters a_{ij} lie on the variety $b_{123} = 0$ ($d = 3$) and more generally on the intersection of varieties $\cap_{i<j<k}(b_{ijk} = 0)$. We may restrict to this variety by putting

$$a_{ij} = \frac{[x_i - x_j - 1]}{[x_i - x_j]} \tag{30}$$

for some new set of free parameters $\{x_1, \ldots, x_d\}$. Other restrictions involve the vanishing of one or more b_{ij} (consider $b_{12} = b_{13} = 0$ in b_{123})—see [29]. For $|q| \neq 1$ this can be achieved from equation (30) by taking a large $|x_i - x_j|$ limit. *Henceforward we will restrict attention to the algebras whose parameters are obtained as in equation (30)—denoted* $\mathsf{D}_n^d = \mathsf{D}_n^d(q, a)$, *or* $\mathsf{D}_n^d(q, x)$ *as in definition 1.* Note that this requires care in the choice of ground ring.

For each $k \in \{1, \ldots, d\}$ there is a natural surjection $\mathsf{D}_n^d \xrightarrow{k} \mathsf{D}_n^{d-1}$ given by $v(k) \mapsto 0$.

Proposition 7. *If B is a basis for D_{n-1}^d then $B_n^d(B)$ is a basis for D_n^d.*

Proof. It is enough to show that B_n^d is a basis of D_n^d. We have so far shown that B_n^d spans D_n^d. Thus (formally) rank $(\mathsf{D}_n^d) \leqslant |B_n^d| = n!d^n$. Suppose now that there is a linear dependence in B_n^d. This would also show up over any ring containing the ground ring (such as the field of fractions). The next section will conclude this proof by showing that there can be no such linear dependence (it will show that the dimension over such a field is at least $n!d^n$). □

3. Generic representation theory

We first review *outer products* of S_n [13, 21]. A *tableau* of shape $\lambda \in \Gamma_n^d$ is any arrangement of the 'symbols' $1, 2, \dots, n$ in the n boxes of λ, and a tableau is standard if each component tableau λ^i is standard. We will denote the set of standard tableaux of shape λ by

$$T^\lambda = \{T_1^\lambda, T_2^\lambda, \dots, T_k^\lambda, \dots\}.$$

We define an order on rows of λ by placing the whole of component diagram λ^{i+1} under λ^i. We then define an order $<$ on standard tableaux of shape λ by

$$T_i^\lambda < T_j^\lambda$$

if the highest number to appear on a different row is in an earlier row in T_j^λ.

Let T_p^λ be a tableau. We define $\sigma_i(T_p^\lambda)$ as the tableau obtained by interchanging 'symbols' i and $i+1$. Note that this action does not necessarily take a standard tableau to a standard tableau. We define $l_{ij}(T_p^\lambda) = -l_{ji}(T_p^\lambda)$ as the reciprocal of the signed hook length h_{ij} from i to j in T_p^λ, putting $l_{ij}(T_p^\lambda) = 0$ if i and j are in different components.

Let R^λ be the space spanned by T^λ. If μ is a diagram (i.e. a d-tuple with $d = 1$) then under a suitable action R^μ is a simple module for S_n associated to that diagram. We write $\dim(\mu) = \dim(R^\mu)$ for the dimensions of these simple modules.

Proposition 8. *The set T^λ is a basis for the left S_n module $S_n(\otimes_{i=1}^d R^{\lambda^i})$ (the outer product $\lambda^1 \otimes \lambda^2 \otimes \dots \otimes \lambda^d$) with action*

$$\sigma_i T_p^\lambda = T_p^\lambda \qquad i, i+1 \text{ in same row of } T_p^\lambda \tag{31}$$

$$\sigma_i T_p^\lambda = -T_p^\lambda \qquad i, i+1 \text{ in same column} \tag{32}$$

and if $\sigma_i(T_p^\lambda)$ is standard (note, this covers all remaining cases) and $t = l_{i\,i+1}(T_p^\lambda)$

$$\sigma_i T_p^\lambda = \mp t T_p^\lambda + (1 \pm t)\sigma_i(T_p^\lambda) \qquad i \text{ in a lower/higher row than } i+1 \text{ in } T_p^\lambda. \tag{33}$$

This is a standard result (see [21] and references therein). Motivated by consideration of a generalized Andrews–Baxter–Forrester model, a suitable generalization of proposition 8 to generic H_n, in case all λ^i are single row diagrams, was given in [31]. Fix x, a d-tuple of complex numbers, and define h_{ij}^x the *generalized hook length* in T_p^λ by

$$h_{ij}^x = h_{ij}^0 + x_i - x_j$$

where h_{ij}^0 is the hook length obtained by superimposing the diagrams containing i and j (see also [32, p 244]). The idea is to replace the hook length by the generalized hook length. In fact this works for arbitrary λ, shown as follows.

Proposition 9. *T^λ is a basis for a left H_n-module R^λ with action*

$$g_i T_p^\lambda = T_p^\lambda \qquad i, i+1 \text{ in the same row of } T_p^\lambda \tag{34}$$

$$g_i T_p^\lambda = -q^2 T_p^\lambda \qquad i, i+1 \text{ in the same column} \tag{35}$$

and if $\sigma_i(T_p^\lambda)$ is standard and $h = h_{i\,i+1}^x$

$$g_i \begin{pmatrix} T_p^\lambda \\ \sigma_i(T_p^\lambda) \end{pmatrix} = \begin{pmatrix} \frac{[h]-q[h+1]}{[h]} & \frac{-q[h-1]}{[h]} \\ \frac{-q[h+1]}{[h]} & \frac{[h]-q[h-1]}{[h]} \end{pmatrix} \begin{pmatrix} T_p^\lambda \\ \sigma_i(T_p^\lambda) \end{pmatrix} \qquad T_p^\lambda < \sigma_i(T_p^\lambda). \tag{36}$$

Proof. The quadratic and commuting relations are readily checked. For the braid relation there are various cases to check by explicit calculation. We start with the case in which i, $i+1$, $i+2$ are each in different parts in T^λ_p. Define $r_\pm(h) = \frac{q[h\pm1]}{[h]}$. A direct calculation shows that

$$R(g_1) = \begin{pmatrix} 1-r_+(h_{12}) & -r_-(h_{12}) & 0 & & & \\ -r_+(h_{12}) & 1-r_-(h_{12}) & 0 & & & \\ 0 & 0 & 1-r_+(h_{13}) & -r_-(h_{13}) & 0 & \\ 0 & 0 & -r_+(h_{13}) & 1-r_-(h_{13}) & 0 & \\ 0 & 0 & 0 & 0 & 1-r_+(h_{23}) & -r_-(h_{23}) \\ 0 & 0 & 0 & 0 & -r_+(h_{23}) & 1-r_-(h_{23}) \end{pmatrix}$$

$$R(g_2) = \begin{pmatrix} 1-r_+(h_{23}) & 0 & -r_-(h_{23}) & 0 & & \\ 0 & 1-r_+(h_{13}) & 0 & 0 & -r_-(h_{13}) & 0 \\ -r_+(h_{23}) & 0 & 1-r_-(h_{23}) & 0 & & \\ 0 & 0 & 0 & 1-r_+(h_{12}) & 0 & -r_-(h_{12}) \\ 0 & -r_+(h_{13}) & 0 & 0 & 1-r_-(h_{13}) & 0 \\ 0 & 0 & 0 & -r_+(h_{12}) & 0 & 1-r_-(h_{12}) \end{pmatrix}$$

give a representation of H_3 for *any* h_{12}, h_{13}, provided that

$$h_{23} = h_{13} - h_{12}. \tag{37}$$

Now consider any T^λ_p and some i such that i is in an earlier part than $i+1$, and $i+1$ is in an earlier part than $i+2$ in T^λ_p. The actions of g_i, g_{i+1} on T^λ block diagonalize, with a typical block of the form $\{T^\lambda_p, \sigma_i(T^\lambda_p), \sigma_{i+1}(T^\lambda_p), \sigma_i(\sigma_{i+1}(T^\lambda_p)), \sigma_{i+1}(\sigma_i(T^\lambda_p)), \sigma_{i+1}(\sigma_i(\sigma_{i+1}(T^\lambda_p)))\}$. Note that the matrices above describe the action of g_i, g_{i+1} (resp.) provided $h_{12} = h^x_{i\,i+1}$, $h_{13} = h^x_{i\,i+2}$ and $h_{23} = h^x_{i+1\,i+2}$—the hook lengths in T^λ_p. Since these obey equation (37) we have a representation of H_n for any choice of x which avoids generalized hooks of length zero. Provided all the diagrams are standard this calculation also works in case fewer parts are involved.

To check cases involving i, $i+1$ in the same row or column we note that our calculation verifies a representation in case all tableaux (not just standard) are used, except that there can be some zero divides. To see this consider the case of i, $i+1$ adjacent in the same row. Then $h^x_{i\,i+1} = 1$ and the representation may be decoupled into a part involving the standard tableau and one involving the non-standard. Nonetheless it is still a representation, so the proof is complete. □

Proposition 10. *The action of* $v(i)$ *given by*

$$v(i)T^\lambda_p = \begin{cases} T^\lambda_p & \text{symbol 1 appears in } \lambda^i \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

equips R^λ *with the property of* D^d_n*-module, with parameters determined by* x.

Proof. This is another set of explicit calculations. The commutation relations are readily checked, so we are left with equation (28). This is easily checked on restriction to D^d_2. □

Note that the representations of $C_d \wr S_n$ in propositions 8 and 10 are recovered in the limit of all the differences $x_i - x_j$ large.

Note that $T^\lambda \cong \cup_{e^j_i} T^{\lambda - e^j_i}$, where $\lambda - e^j_i$ denotes removing a box from the ith row of the jth part, and the sum is over pairs (i, j) such that $\lambda - e^j_i \in \Gamma^d_{n-1}$. Thus we have the following proposition.

Proposition 11. *The generic R^λ restriction rules for $\mathsf{D}^d_{n-1} \subset \mathsf{D}^d_n$ are*

$$\mathrm{Res}^n_{n-1}(R^\lambda) \cong \bigoplus_{e^j_i} R^{\lambda - e^j_i}. \tag{39}$$

For example, $R^{((2^2),(1))} \cong R^{((2,1),(1))} \oplus R^{((2^2),\emptyset)}$ as a D^2_4-module.

There follow two explicit examples of the action on T^λ described in propositions 8 and 10. Firstly, with $\lambda = ((1),(1),(1))$ we may take $R(g_i)$ as in equation (37) and

$$R(v(1)) = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} R(v(2)) = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R(v(3)) = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Secondly, with $\lambda = ((1),(2))$ we have

$$R(g_1) = \begin{pmatrix} 1 - r_+(h) & -r_-(h) & 0 \\ -r_+(h) & 1 - r_-(h) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R(g_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - r_+(h+1) & -r_-(h+1) \\ 0 & -r_+(h+1) & 1 - r_-(h+1) \end{pmatrix}$$

where $h = x_1 - x_2$, and

$$v(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad v(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this is a representation of D^2_3 for a choice of the a_{ij} deformation parameters determined by x. By restriction to D^2_2 and comparison with the representation in equation (24) we obtain $1 - qa_{12} = 1 - r_+(h)$ and hence $a_{12} = \frac{[x_1 - x_2 + 1]}{[x_1 - x_2]}$.

Proposition 12. *For generic parameters D^d_n is semisimple. The sets T^λ for all $\lambda \in \Gamma^d_n$ form bases for a complete set of unequivalent irreducible representations of D^d_n.*

Outline proof. The proof is an induction, following [6]. Suppose that the proposition is true at level $n-1$. Since no two R^λs restrict to the same sum of irreducibles at this level then they are distinct. By proposition 22 (in the appendix) they are also simple. This argument holds in particular in the case of $C_d \wr S_n$, where we have already shown this number of simples to be maximal. It follows that the total contribution to the dimension of the algebra coming from these simples in general is $d^n n!$, as there. This saturates the bound already established for proposition 7, and the proof is complete. □

A useful combinatorial analysis of these representations is given in appendix B.

3.1. *On primitive idempotents*

The *q-(anti)symmetrizers* (the primitive and central idempotents of $H_n(q)$) are $y^{(1^n)} = (q^{\binom{n}{2}}[n]!)^{-1} y_u^{(1^n)}$ and $y^{(n)} = (q^{\binom{n}{2}}[n]!)^{-1} y_u^{(n)}$ [34] where

$$y_u^{(1^n)} = -(-1)^{\frac{n(n-1)}{2}} \sum_{w \in B_n^{\mathrm{red}}} (-1)^{-\mathrm{len}(w)} w \qquad y_u^{(n)} = (q^2)^{\frac{n(n-1)}{2}} \sum_{w \in B_n^{\mathrm{red}}} (q^2)^{-\mathrm{len}(w)} w$$

and $q^{\binom{n}{2}}[n]! = \sum_w (q^2)^{\mathrm{len}(w)}$. We have

$$g_i y^{(n)} = y^{(n)} \qquad g_i y^{(1^n)} = -q^2 y^{(1^n)} \qquad (y^{(n)})^T = y^{(n)} \qquad (y^{(1^n)})^T = y^{(1^n)}. \tag{40}$$

Put $\varepsilon_\pm = (iq)^{1\mp 1}$, and let $y_u^\pm, y^\pm$ be the unnormalized and normalized $n = 2$ q-(anti)symmetrizers:

$$y_u^+ = y_u^{(2)} = q^2 + g_1 \qquad y^+ = y^{(2)} = \frac{q^2 + g_1}{q^2 + 1}$$

$$y_u^- = y_u^{(1^2)} = 1 - g_1 \qquad y^- = y^{(1^2)} = \frac{1 - g_1}{q^2 + 1}.$$

Another basis of D_2^d (cf B_2^d) is

$$B_2'^d = \{v(i)g_1v(j), (g_1 + q^2)v(i)g_1v(j)|i, j = 1, 2, \ldots, d\}. \tag{41}$$

Indeed, a manifestly generically simple right D_2^d-submodule $S^\lambda = y_u^+ v(12)\mathsf{D}_2^d$ of D_2^d is spanned by

$$B^{+ij} = \{(g_1 + q^2)v(i)g_1v(j)|\{i, j\} = \{1, 2\}\}$$

and similarly for each distinct pair $\{i, j\}$. Modulo these, $\{(g_1 + q^2)v(i)g_1v(i)\}$ spans a rank 1 submodule for each i. We may associate primitive and central idempotents to these modules.

Proposition 13. *The primitive and central idempotents of* D_2^d *are*

$$Y_j^\pm = \varepsilon_\pm^{-1} v(j)g_1\left(v(j) + \sum_{i\neq j} \frac{b_{ji}}{b_{ji} - \varepsilon_\pm} v(i)\right)y^\pm \qquad (j = 1, 2, \ldots, d).$$

We also have

$$Y_j^+ = v(j)g_1\left(1 + \sum_{i\neq j} \frac{v(i)}{b_{ji} - 1}\right)y^+ = \left(v(j) + \sum_{i\neq j} \frac{v(j)g_1v(i)}{b_{ji} - 1}\right)y^+$$

$$= v(j)\left(g_1 + \sum_{i\neq j} \frac{g_1v(i)g_1}{b_{ji} - 1}\right)y^+.$$

Note that $y_u^{(2)}\mathsf{D}_2^d y_u^{(1^2)}$ is spanned by $\{y_u^{(2)}v(i)g_1v(j)y_u^{(1^2)}|i < j\}$, and hence that

$$y^{(2)}\mathsf{D}_2^d Y_k^- = 0. \tag{42}$$

Finally in this section we note that the restriction rules for the D_n^d-modules R^λ under the restriction $\mathsf{D}_n^d \subset H_n$ are known, since these modules are the standard outer product representations on specialization to S_n [21].

4. Quotient algebras with large n limits

Let $\overline{\Gamma}_n^d$ be the subset of Γ_n^d in which each λ has each λ^i a *single row diagram* (or empty). For $k = 1, 2, \ldots, d$ we write $\psi_k(\overline{\Gamma}_n^{d-1})$ for the subset of $\overline{\Gamma}_n^d$ obtained by inserting an empty part at position k in each $\lambda \in \overline{\Gamma}_n^{d-1}$.

Definition 6. *We define quotient algebra* $\overline{\mathsf{D}}_n^d$ *by a short exact sequence*

$$0 \to \sum_j I_j \to \mathsf{D}_n^d \to \overline{\mathsf{D}}_n^d \to 0 \tag{43}$$

where I_j *is the double-sided ideal generated by* Y_j^-.

Proposition 14 follows from the restriction rules in equation (39).

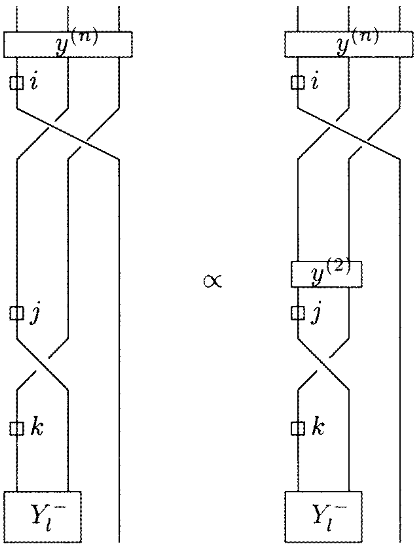
Proposition 14. *Generically, $\overline{\mathsf{D}}^d_n$ is semisimple, with irreducible representations indexed by $\overline{\Gamma}^d_n$.*

We wish to bring this algebra into a form suitable for constructing a 'thermodynamic' limit.

Consider the right D^d_n-module $\mathcal{T}^R_n = y^{(n)}_u \mathsf{D}^d_n$ (let $\mathcal{T}^L_n = \mathsf{D}^d_n y^{(n)}_u$ be the corresponding left module). This has a basis $B^{(n)} = \{y^{(n)}_u v(s) | s$ any signature$\}$, with d^n elements.

Proposition 15. *The D^d_n-module $\mathcal{T}^R_n$ is also a module for the quotient $\overline{\mathsf{D}}^d_n$.*

Proof. Note that $y^{(2)}_u \mathsf{D}^d_2 Y^-_j = K y^{(2)}_u Y^-_j = 0$ and $y^{(n)}_u \mathsf{D}^d_n Y^-_j$ factors to include this, as illustrated in equation (44).



(44)

(We will use variants of this factor trick repeatedly in what follows.) □

We call the module $\mathcal{T}^R_n$ *tensor space* (in certain cases it restricts to the tensor representation of H_n, and it is generically isomorphic to it). We will show that it is a faithful $\overline{\mathsf{D}}^d_n$-module.

Direct calculation shows the useful result (for $i \neq j$, $b_{ij} \neq 1$)

$$y^+ v(ij) y^- = \frac{-q^2 - b_{ij}}{1 - b_{ij}} y^+ v(ji) y^- \tag{45}$$

which, by means similar to equation (44), may be generalized to

$$y^{(n)} v(\ldots ij \ldots) y^{(1^n)} = \frac{-q^2 - b_{ij}}{1 - b_{ij}} y^{(n)} v(\ldots ji \ldots) y^{(1^n)} \tag{46}$$

for $\ldots ij \ldots$ a signature which is a permutation of $\{1, 2, \ldots, n\}$. Define

$$\chi_n = y^{(n)} \left(\sum_{w' \in S_n} (-1)^{\mathrm{len}(w')} v(w') \right)$$

and

$$x_{ij} = \frac{-q^2 - b_{ij}}{1 - b_{ij}}.$$

Proposition 16. *For $w \in S_n$ (a permutation of $\{1, 2, \ldots, n\}$) and $d = n$*

$$y^{(n)} v(w) y^{(1^n)} = (-1)^{\mathrm{len}(w)} k_n \chi_n \tag{47}$$

where k_n is a fixed scalar.

Proof. Similarly to equation (45) we have, for $d = 2$, $\{i, j\} = \{1, 2\}$,

$$y^+ v(ij) y^- \propto y^+(v(12) - v(21)). \tag{48}$$

It follows from equation (46) that there is only one linear combination, X say, required (up to the overall scalar) to express $y^{(n)} v(w) y^{(1^n)}$ for any w. Note that this X must obey $X g_i = -q^2 X$ for all i, and hence that any such (non-zero) X will do. It follows by repeatedly applying a variant of the factor trick, i.e. as in equation (44), using equation (48) that the given form obeys these relations. It then follows from the form of χ_n that the scalar takes the given form. □

Note that S_n acts on the set of signatures of D_n^d by permuting the order of the sequence. The orbits of this action are indexed by the non-decreasing signatures. The *support* of a signature is the set of those symbols appearing at least once. Non-decreasing signatures are the same combinatorial objects as *weights* in Lie theory [23]. Thus the *degree* of a signature will here be the number of elements in the sequence (and hence n), and the *depth* of a signature is the degree of the support (and hence $\leqslant d$). Further, the *dominance order* on weights, given by

$$\lambda \geqslant \mu \qquad \text{if} \quad \sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i \quad \forall j$$

induces a partial order on non-decreasing signatures, which we will also call $\geqslant$. Thus, for example, 1123333 and 1222223 are not comparable.

Suppose that an element of D_n^d can be written in the form $h_1 v(s) h_2$, where h_i is an element of the Hecke subalgebra. This form is said to be an i-degenerate form of the element if the signature s has $n - i$ distinct symbols occurring.

Proposition 17. (i) $y_u^{(d)} \mathsf{D}_d^d y_u^{(1^d)} = K \chi_{n=d}$. (ii) *For $n > d$*, $y_u^{(n)} \mathsf{D}_n^d y^{(1^{d+1})} = 0$.

Proof. (i) Note firstly that the left-hand side contains the right (we have equality if we restrict on the left to non-degenerate signatures). We will work inductively on the number of distinct symbols in a signature (starting from d and reducing). The base can be d itself, but to illustrate the calculations involved we will rather take $d - 1$. In this case we proceed as follows. For $w = 12 \ldots d - 1$ let w_j^k denote the insertion of symbol j in the kth position in this sequence. It follows that $(-q^{-2})^{k-1} X_k = (-q^{-2})^{k'-1} X_{k'}$, for all k, k', where

$$X_k = \sum_{j=1}^{d} y^{(n)} v(w_j^k) y^{(1^n)}. \tag{49}$$

Taking into account equation (46) these summands may be brought into a form with signature a non-decreasing sequence, and hence there are at most d of them independent. Varying k, k' we obtain $d - 1$ nominally independent linear equations here for elements of the form $y^{(n)} v(123345 \ldots d - 1) y^{(1^n)}$, in terms of $y^{(n)} v(12345 \ldots d - 1d) y^{(1^n)}$. Regarding the last of these as given, we require to show that the nullity of a certain coefficient matrix is zero. Specifically, taking the linear equations to be $X_k + q^{-2} X_{k+1} = 0$ we must check that

$$\left| \begin{pmatrix} 1 + q^{-2} & x_{21} + q^{-2} & x_{32}(x_{31} + q^{-2}) & x_{42} x_{43}(x_{41} + q^{-2}) & \cdots \\ 1 + x_{21} q^{-2} & 1 + q^{-2} & x_{32} + q^{-2} & x_{43}(x_{42} + q^{-2}) & \cdots \\ x_{21}(1 + x_{31} q^{-2}) & 1 + x_{32} q^{-2} & 1 + q^{-2} & x_{43} + q^{-2} & \cdots \\ \cdots & & & & \end{pmatrix} \right|$$

is non-zero. This is a manifestly non-zero algebraic equation in the x_{ij} (consider the case of all x_{ij} small for $i > j$), so *generically* we are done for signatures with support $\{1, \ldots, d-1\}$. Other supports of degree $d-1$ follow similarly, so we are done at level $d-1$.

For level $d-i$ consider a sum as above in which i symbols are inserted into $w = 12\ldots d-i$ (insertions into other non-degenerate subsequences will again work similarly). Here we have

$$\sum_{j_1=1}^{d} \cdots \sum_{j_i=1}^{d} y^{(n)} v(w^{k_1\ldots k_i}_{j_1\ldots j_i}) y^{(1^n)} = 0$$

(provided $i > 1$) by an obvious generalization of the above scheme. The number of distinct symbols in the signature of each summand is by construction at least $d-i$. By inductive assumption we may collect together all summands with $d-(i-1)$ or more symbols as given. The remainder can be characterized by Young diagrams of degree $n = d$ and depth exactly $d-i$. Following the argument above put $x_{ij} = 0$ for all $i > j$. Then we can ignore the summands which do not have non-decreasing signatures. The number of summands still remaining is $\begin{pmatrix} d-1 \\ d-i-1 \end{pmatrix}$, so it is sufficient to show that there are at least this many independent constraints.

Consider the subset of constraints constructed in correspondence with our set of i-degenerate non-decreasing signatures as follows. For each signature replace all but the last occurence of each symbol by a variable j_l (some l). Now order these constraints in an order consistent with the $\leqslant$ order on the corresponding signatures. It is straightforward to see that the matrix of coefficients of i-degenerate elements for this set of constraints is lower uni-triangular.

(ii) Follows immediately. $\square$

For $x = \sum_i k_i v(i)$ and w_1, w_2 sequences (possibly empty) let $v(w_1 x w_2) = \sum_i k_i v(w_1 i w_2)$. For example, $v(12x54) = \sum_i k_i v(12i54)$. Then

$$y^+ v(11) Y_1^+ = y^+ v(\Sigma_1 1)$$

where $\Sigma_i = \Sigma(v(i))$ and Σ is the linear transformation on A given by

$$\Sigma(v(i)) = v(i) + \sum_{j \neq i} \frac{b_{ij}}{b_{ij} - 1} v(j).$$

Let $\Sigma_i^{(l)} = \Sigma^l(v(i))$ (i.e. $\Sigma^{(2)} = \Sigma \circ \Sigma$, and so on). Then, similarly,

$$y^{(3)} v(111) Y_1^{(3)} = y^{(3)} v(\Sigma_1^{(2)} \Sigma_1 1)$$

and

$$y^{(n)} v(11\ldots 11) Y_1^{(n)} = y^{(n)} v(\Sigma_1^{(n-1)} \Sigma_1^{(n-2)} \ldots \Sigma_1 1).$$

We will write $v_r[11\ldots 11]$ for $v(\Sigma_1^{(n-1)} \Sigma_1^{(n-2)} \cdots \Sigma_1 1)$. More generally, if w is a word in the symbols $1, 2, \ldots, d$ containing μ_1 1's, μ_2 2's and so on (of *weight* $\mu = (\mu_1, \mu_2, \ldots)$), then we write $v_r[w]$ (resp. $v_l[w]$) for the variation of $v(w)$ in which the ith j in the sequence, counting from the left (resp. right), is replaced by $\Sigma_j^{(i-1)}$. For example,

$$v_r[3332221111] = v(\Sigma_3^{(2)} \Sigma_3 3 \Sigma_2^{(2)} \Sigma_2 2 \Sigma_1^{(3)} \Sigma_1^{(2)} \Sigma_1 1).$$

Proposition 18. *(i) For $n > d$, $y_u^{(1^{d+1})} \in \sum_j \mathsf{D}_n^d Y_j^- \mathsf{D}_n^d$. (ii) Tensor space is generically faithful, specifically*

$$y^{(n)} \mathsf{D}_n^d \cong \bigoplus_\lambda R^\lambda.$$

Proof. (i) If (ii) is true then this follows from proposition 17(ii). (ii) Consider the following useful identities, obtained by direct calculation. For $i \neq j$

$$y^{(2)}v(i\Sigma_j)g_1 = r^1_{ji}y^{(2)}v(i\Sigma_j) + (1 - r^1_{ji})y^{(2)}v(\Sigma_j i)$$

where $r^1_{ji} = \frac{-q^2 b_{ji}}{b_{ji}-1}$;

$$y^{(2)}v(i\Sigma_i)g_1 = -q^2 y^{(2)}v(i\Sigma_i) + (1 + q^2)y^{(2)}v(\Sigma_i i)$$
$$y^{(2)}v(ij)g_1 = r^0_{ji}y^{(2)}v(ij) + (1 - r^0_{ji})y^{(2)}v(ji)$$

where $r^0_{ji} = b_{ji}$; and more generally,

$$y^{(2)}v(\Sigma_i^{(i)}\Sigma_j^{(m)})g_1 = r^{m-l}_{ji}y^{(2)}v(\Sigma_i^{(l)}\Sigma_j^{(m)}) + (1 - r^{m-l}_{ji})y^{(2)}v(\Sigma_j^{(m)}\Sigma_i^{(l)})$$

where $r^m_{ji} = 1 - q\frac{[x_j - x_i + m + 1]}{[x_j - x_i + m]}$.

It follows from these identities, by repeated application of the factor trick, that the submodule of tensor space generated by, for example,

$$y^{(10)}v(\Sigma_3^{(2)}\Sigma_3 3\Sigma_2^{(2)}\Sigma_2 2\Sigma_1^{(3)}\Sigma_1^{(2)}\Sigma_1 1)$$

is spanned by elements of the same form in which two components $\Sigma_i^{(l)}\Sigma_j^{(m)}$ in the generalized signature may be interchanged if $i > j$. It follows similarly that this module is R^λ—here in the case where $\lambda = ((4), (3), (3))$. Thus, at least generically, every irreducible representation of $\overline{\mathsf{D}}^d_n$ occurs as a submodule of tensor space (in fact a dimension count shows that each one occurs exactly once). □

It follows that the idempotent $y^{(n)}$ is a sum of one representative primitive idempotent from each irreducible class. Thus if $y^{(n)}v_\lambda\overline{\mathsf{D}}^d_n = y^{(n)}v_\lambda\mathsf{D}^d_n$ is simple, i.e. is R^λ say, for some $v_\lambda \in \mathsf{D}^d_n$, then v_λ kills all but one of these primitive idempotents, and hence $\overline{\mathsf{D}}^d_n y^{(n)}v_\lambda$ is the corresponding simple *left* module. Note that suitable choices for the v_λs here are the generalized signature elements $v_r[w]$ of generalized signature weight λ. Thus $v_l[w]y^{(n)}v_r[w'] = 0$ if w, w' have different weights. It also follows generically that $\overline{\mathsf{D}}^d_n y^{(n)}\overline{\mathsf{D}}^d_n$ is the regular representation. Thus

$$\overline{\mathsf{D}}^d_n = \overline{\mathsf{D}}^d_n y^{(n)}\overline{\mathsf{D}}^d_n = \sum_\lambda \overline{\mathsf{D}}^d_n y^{(n)}\overline{\mathsf{D}}^d_n = \sum_{\lambda\in\overline{\Gamma}^d_n}\sum_{w\in T^\lambda}\sum_{w'\in T^\lambda} K v_l[w]y^{(n)}v_r[w'] \qquad (50)$$

is a decomposition of $\overline{\mathsf{D}}^d_n$ into a natural basis exhibiting the matrix structure.

Recall that $(y_u^{(1^d)})^{(n)}$ denotes $y_u^{(1^d)}$ acting on strings $n+1$ to $n+d$. Thus $[(y_u^{(1^d)})^{(n)}, \mathsf{D}^d_n] = 0$ and $(y_u^{(1^d)})^{(n)}\,\mathsf{D}^d_{n+d}$ is a left D^d_n-right D^d_{n+d}-module.

Accordingly, for each n, d define functors on categories of left modules

$$\mathcal{F} : \overline{\mathsf{D}}^d_{n+d} - \text{mod} \to \overline{\mathsf{D}}^d_n - \text{mod}$$
$$\mathcal{F} : M \mapsto (y_u^{(1^d)})^{(n)}M$$
$$\mathcal{G} : \overline{\mathsf{D}}^d_n - \text{mod} \to \overline{\mathsf{D}}^d_{n+d} - \text{mod}$$
$$\mathcal{G} : N \mapsto \overline{\mathsf{D}}^d_{n+d}(y_u^{(1^d)})^{(n)} \otimes_{\overline{\mathsf{D}}^d_n} N.$$

Define right module functors similarly. The action of $\mathcal{G}$ on a module of R^λ type is illustrated in figure 3. The large (dotted) box at the top symbolizes a general element of $\overline{\mathsf{D}}^d_{n+d}$ in the decomposed regular representation formulation of equation (50).

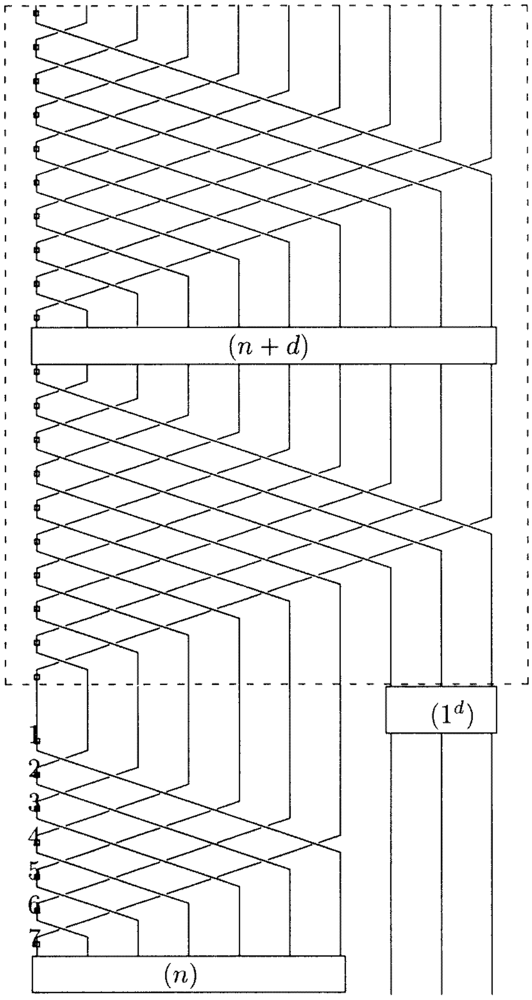


Figure 3. Illustration of the action of $\mathcal{G}$ on a module of R^λ type. A box across n strings with label (λ) denotes $y_n^{(\lambda)}$.

Proposition 19. *For* $[d]! \neq 0$ *there is an isomorphism of unital algebras*

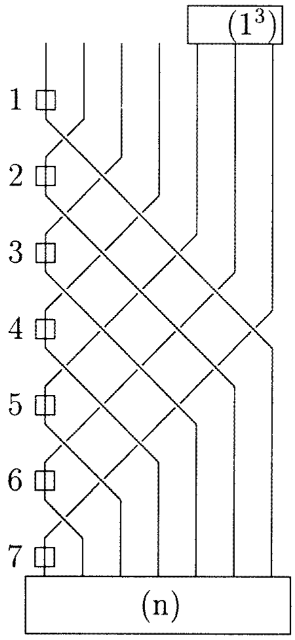
$$\Omega : y_u^{(1^d)} \overline{\mathsf{D}}_{n+d}^d y_u^{(1^d)} \xrightarrow{\sim} \overline{\mathsf{D}}_n^d \tag{51}$$

and

$$\mathcal{F}(R^\lambda) \cong \begin{cases} R^{\lambda-(1^d)} & \lambda - (1^d) \in \Gamma \\ 0 & \textit{otherwise} \end{cases} \tag{52}$$

$$\mathcal{G}(R^\lambda) \cong R^{\lambda+(1^d)}. \tag{53}$$

Proof. Equation (51) follows generically from equation (52). The proof of equation (52) is well illustrated by the following diagram. Note, in particular, that by the invertibility of g_i we may apply the factor $y_u^{(1^d)}$ anywhere along the 'top' of the diagram.



It will suffice to consider three cases for the generalized signature. Firstly, note that if $\Sigma_i^{(l)}$ is 'higher' in the diagram than $\Sigma_i^{(m)}$ then $l < m$, so, noting that we have a *generalized* signature, we need then only record the i (that is the subscript index) for each term, since the other details may be recovered unambiguously. In this formalism consider generalized signature 1112233 in the diagram. We see, on 'commuting through', that the diagram is zero, using the identity $y^{(1^2)}y^{(2)} = 0$. Next consider generalized signature 1112323. The action of $y^{(1^3)}$ on the last three elements of the signature gives zero, since $R^{(1,2)}(y^{(1^3)}) = 0$. Finally, consider generalized signature 1123123. We claim that the isomorphism maps this to 1123 (that is to say, the corresponding basis element at $n = 4$). This gives a well-defined map by restriction. It is an isomorphism by proposition 17(i) (here in the case $d = 3$, but the general principle will be clear).

The proof of equation (53) is illustrated by the diagrammatic expression in figure 3. The action of $\mathcal{G}$ is illustrated on some left module R^λ (the lower part of the diagram). The presence of the $y^{(1^d)}$ ensures that the part of the signature element between this and $y^{(n+d)}$ simply involves a (partial) signature which is a permutation of $12\ldots d$. The corollary to proposition 18 then tells us that the left module built in this way is as claimed; i.e., the signature weights above and below the $y^{(n+d)}$ must coincide. □

The generic structure of $\overline{\mathsf{D}}_n^d$ may be verified immediately from this (by iteration on n, d) on noting the following.

Proposition 20. *The algebra* $\overline{\mathsf{D}}_{n+d}^{[d-1]}$ *defined by the short exact sequence*

$$0 \to \overline{\mathsf{D}}_{n+d}^d y_u^{(1^d)} \overline{\mathsf{D}}_{n+d}^d \to \overline{\mathsf{D}}_{n+d}^d \to \overline{\mathsf{D}}_{n+d}^{[d-1]} \to 0$$

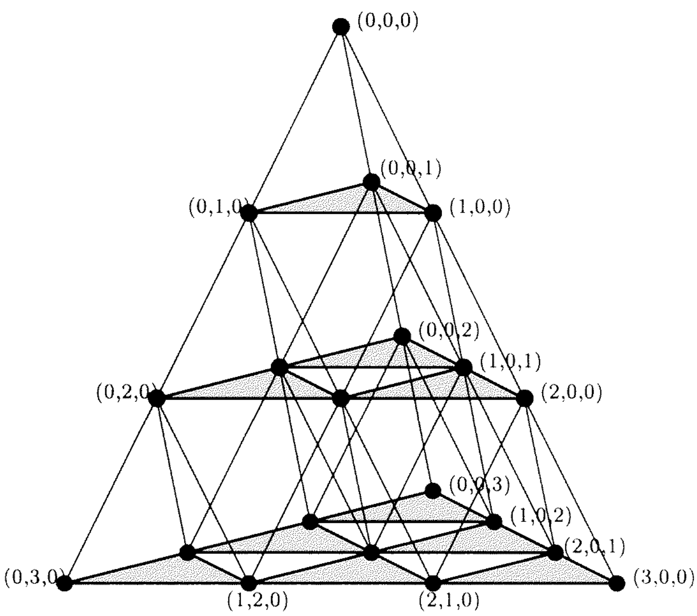
has simple modules indexed by $\cup_{k=1}^d \psi_k(\overline{\Gamma}_{n+d}^{d-1})$.

5. Results and applications

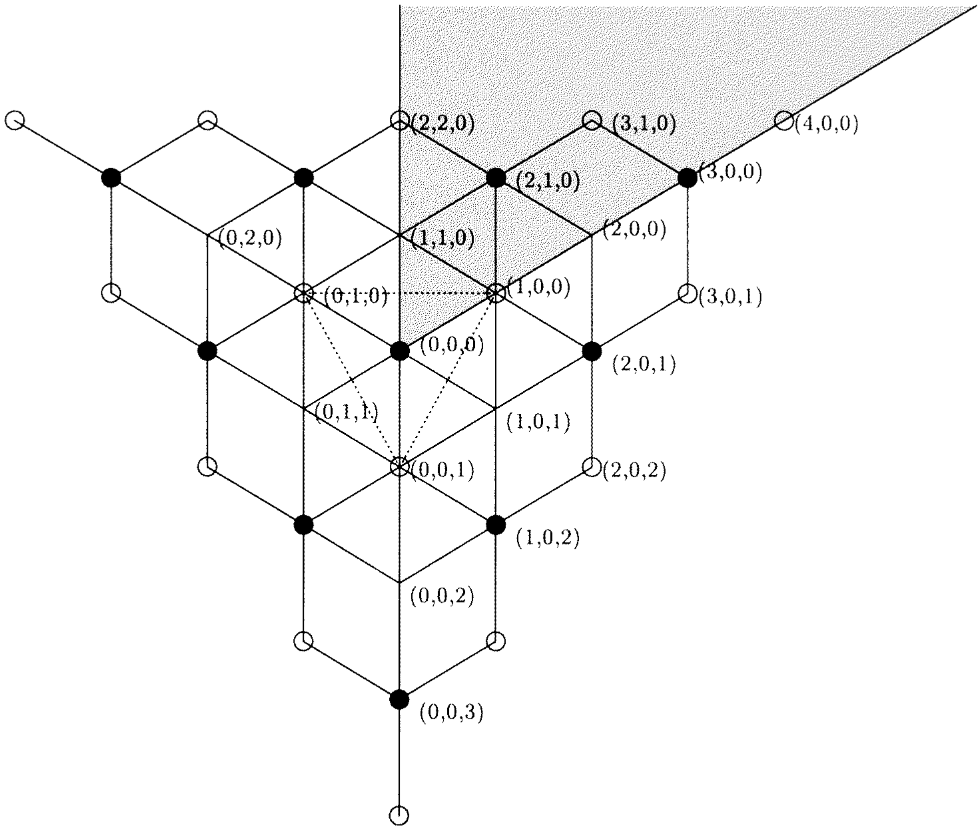
Put $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *weight* λ of depth N is an element of $\mathbb{N}^N$. The degree of λ is $|\lambda| = \sum_i \lambda_i$. There is an action of S_N on the set of weights, permuting the indices i. The orbits of this action may be indexed by the *dominant weights* (those with $\lambda_i \geqslant \lambda_{i+1}$). Recall that the ordinary Hecke algebra quotient H_n^N has irreducible representations indexed by the set of dominant weights of depth N and degree n. The global (large n) version H_-^N of H_n^N [34] has irreducible representations indexed by the set Λ_-^{N-1} of all dominant weights of depth $N-1$. Obviously $\mathbb{N}^N$ embeds in the vector space $\mathbb{R}^N$, and the set Λ_-^{N-1} is isomorphic to the set $\Lambda_-^N/(1, 1, \ldots, 1)$ of dominant weights of depth N modulo the vector $(1, 1, \ldots, 1)$.

Proposition 21. *The algebra $\overline{\mathsf{D}}_n^d$ has irreducible representations indexed by the set of weights of depth d and degree n. The correspondence with the index set given in proposition 14 is $(\lambda_1, \lambda_2, \ldots) \mapsto ((\lambda_1), (\lambda_2), \ldots)$. The global algebra $\overline{\mathsf{D}}_*^d$ has irreducible representations indexed by $\mathbb{N}^d/(1, 1, \ldots, 1)$.*

For example, the following picture shows layers indexing the simple modules of (respectively, from the top down) $\overline{\mathsf{D}}_0^3 \subset \overline{\mathsf{D}}_1^3 \subset \overline{\mathsf{D}}_2^3 \subset \overline{\mathsf{D}}_3^3 \subset \cdots$. The horizontal connecting lines within layers are here as a guide to the eye only; the lines *between* layers indicate the induction/restriction rules between simple modules.



Now looking down from above in the $(1, 1, 1)$ direction (so that points $(0, 0, 0)$ and $(1, 1, 1)$ coincide), we obtain the weight space for the global algebra. A part of this (close to the origin) is shown below.

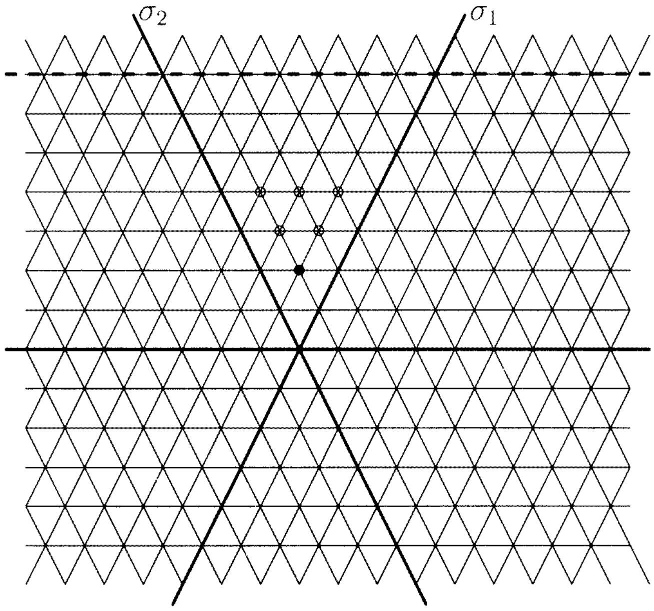


This is the usual A_2 weight picture [23]. The dominant region is shaded. Each weight (vertex) corresponds to a fibre of representations generated by the action of the $\mathcal{F}/\mathcal{G}$ functors. For example, (0, 0, 0) corresponds to the representations (0, 0, 0) of $\overline{\mathsf{D}}^3_0$, (1, 1, 1) of $\overline{\mathsf{D}}^3_3$, (2, 2, 2) of $\overline{\mathsf{D}}^3_6$, and so on. In general, considering an associated physical Hamiltonian on a sequence of different lattice sizes approaching the thermodynamic limit, a given fibre will pick out the same part of the spectrum (that is to say, corresponding to the same physical observable) at each lattice size (cf [34]).

The utility of this picture is particularly striking when one considers the representation theory for cases in which one or more $a_{i\,i+1}$ may be written in the form $\frac{[m-1]}{[m]}$ where m is a positive integer. (The validity of the picture in such a case is the subject of a separate paper, but for the moment let us take it as read.) If l parameters may be written in this way we call it an *l-fold critical case*. Put $m_i = x_i - x_{i+1} (i = 1, \ldots, d, x_{d+1} = x_1)$ so that $a_{i\,i+1} = \frac{[m_i-1]}{[m_i]}$. Any $d-1$ of the m_is may be chosen as positive integers, but since $\sum_i m_i = 0$ the remaining one (m_d, say) may not. However, if q is an rth root of unity then $[m] = [m+r]$, so then a d-fold critical case is possible, with the m in a_{d1} given by $m'_d = m_d + kr > 0$ (some k). Consider the reflection hyperplanes in $\mathbb{N}^d/(1, 1, \ldots, 1)$ which have the effect of permuting adjacent coefficients (and collectively generating an action of S_d). One draws l *affine* reflection hyperplanes (in our example, lines) on the diagram each at a distance m_i (resp. m'_d) from the origin. If $l < d-1$ we generate a subgroup of the A_{d-1} Weyl group S_d in this way. If $l = d-1$ we generate the Weyl group, and if $l = d$ we generate the (infinite) affine Weyl group. In any case the closure of the set of hyperplanes under their own reflecting action gives us a partition of the space into *alcoves, walls* and so on, as described in [35] (these parts are collectively called *facets*). Each part will contain some number of weights. The weights in the closure of the fundamental alcove are representatives of the orbits of the (affine) Weyl group action.

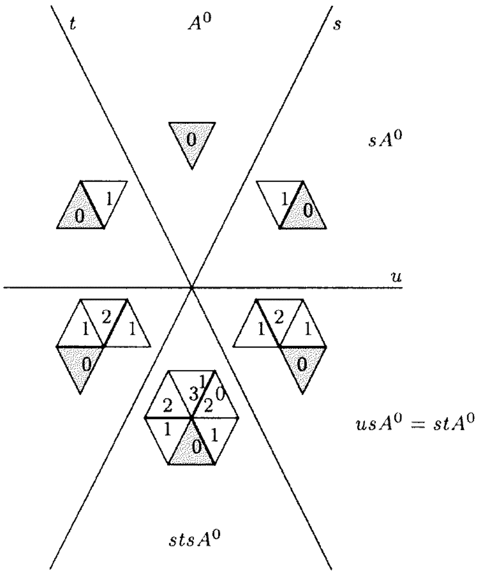
Recall that each weight corresponds to a Specht module, and hence to a simple module (the head of the Specht module), and hence to a part of the physical spectrum [33] in any corresponding q-spin chain. The point of physical interest here is that Specht modules are generically simple, so a non-trivial decomposition of a Specht module into simple modules at some q value signals an increase in Hamiltonian spectrum degeneracy at that q value (as in a spectrum level crossing [33]) in any corresponding q-spin chain. The content of this decomposition indicates *which parts* of the spectrum are coming together. Extending from [35] we may firstly assume that there is no q-level crossing of this kind between two Hamiltonian eigenvalues unless they are associated to weights in the same (affine) Weyl group orbit. This is an extension of the *linkage principle* [25]. Secondly, extending [43] we can say which weights within an orbit will produce a crossing. In the thermodynamic limit this data depends on the orbit only through the type of facet it involves. Here we will describe how to recover the data for weights in the alcoves themselves.

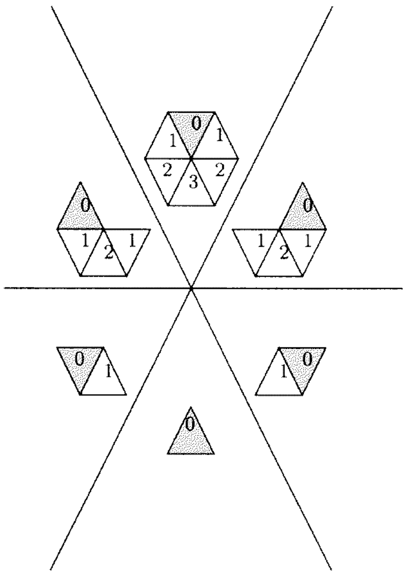
Rotating the picture above through 30° (to make contact with the pictures in [35]) and somewhat expanding the region covered, we have an A_2 *alcove diagram* as follows.



In this picture the case $m_1 = m_2 = 1$ is shown (note, this is $a_{12} = a_{23} = 0$). The corresponding affine reflection lines are marked σ_1 and σ_2. The solid horizontal line is the closure of this set (i.e. the reflection of σ_1 in σ_2). The dashed horizontal line shows where the affine m_3' line *would* appear if q were a root of unity, for example with $r = 7$. The solid circle indicates the location of the $(0, 0, 0)$ weight in this picture. Every vertex is a weight, but a few other dominant weights have been marked with circles (namely, from right to left, $(1, 0, 0)$ and $(1, 1, 0)$ on the next horizontal layer, and then $(2, 0, 0)$, $(2, 1, 0)$ and $(2, 2, 0)$ on the next).

We want the *decompositions* of the Specht modules into simple modules. Here decomposition means the multiplicity of each of the various composition factor simple modules in a filtration by such modules (as in [35]). Staying with the above example, i.e. with $d = 3$ and, say, $x = (3, 2, 1)$ (and q not a root of unity—a 2-fold critical case) the decompositions are illustrated in the following diagrams:





The left-hand figure actually shows the decomposition of the *projective modules* into Specht modules [24] (the projective modules' decomposition is of no direct physical interest, but we can use it to determine the Specht modules' content by Brauer–Humphreys reciprocity [18]). The right-hand figure shows the simple module content of the Specht modules. The data are represented as follows. In each figure there are six alcoves (the reflection lines are here marked s, t, u), and within each alcove we have placed a subfigure which determines the factor module content of any module whose weight lies in that alcove. Each subfigure is made up of triangles, each of which represents a factor in the decomposition of the module in question. The triangle representing the (ever present) factor module with the *same* weight is indicated by shading. The attitude of each remaining triangle with respect to this determines the alcove to which the corresponding factor module belongs. Thus for example (on the left) a projective module in the fundamental alcove is isomorphic to the corresponding Specht module, while the Specht module filtration of a projective in the $stsA^0$ alcove contains one Specht module from each of the six alcoves. The numbers in the triangles are part of the calculation whereby these results are determined (a direct generalization of the procedure for H_n^N described in [35, 43]). The key to the right-hand picture is analogous, so for example a Specht module in the stA^0 alcove contains a copy of the corresponding stA^0 simple together with a copy of the $stsA^0$ simple in the same orbit.

A completely concrete example is obtained by working out the dimensions of generic *ordinary* Hecke algebra Specht (simple) modules from this construction. These modules are the simple heads of the Specht modules residing in the dominant alcove (with the same weight). Thus, for example, it is well known that the $(1, 1, 1)$ ordinary Hecke simple ($n = 3$) is one-dimensional. The $(1, 1, 1){\sim}(0, 0, 0)$ Specht module in our generalized case is six-dimensional, but our picture shows that this is composed of one copy of each of the $(0, 0, 0)$, $(1, 0, 2)$, $(0, 2, 1)$, $(0, 0, 3)$, $(0, 3, 3)$ and $(0, 2, 4)$ simple modules. The last two of these have zero dimension at $n = 3$. The $(0, 0, 3)$ module has dimension 1 (walks of length 3 from $(0, 0, 0)$ to $(0, 0, 3)$). The picture shows that the Specht modules for the two components

$(1, 0, 2)$ and $(0, 2, 1)$ (which each have dimension 3) each consist of the simple for that module together with the $(0, 0, 3)$ simple (and some nominal dimension zero components). Thus their simples have dimension $3 - 1 = 2$. Overall the dimension of the $(0, 0, 0)$ simple is thus $6-2-2-1 = 1$ as required. The reader will readily verify that this analysis gives the correct dimensions in complete generality.

5.1. Remarks

It should be possible to analyse representations of D_n^d derived from vertex models and from cabled (fusion) models in terms of the structure determined here (cf [35]). Work on this is in progress. This will facilitate a systematic investigation of the relationship between spectrum and boundary conditions for various quantum spin chains (cf [30, 33]). We have not concentrated on any particular physical system here, and the physical conditions determined by a specific choice of the boundary parameters x depend on the system, but our analysis determines certain properties of the spectrum for all suitable systems. For example, we have shown that there is a choice of boundary parameters for which the exact spectrum of an ordinary open boundary model will appear as a subset of the model spectrum, and also that there is a set of special choices for x for which other open boundary models appear in the same way. (Each of these corresponds roughly to an ABF model [4] in which the heights are bounded below by a different integer, but the Boltzmann weights are also different—this is the topic for a separate paper.)

To make contact between D_n^d and G_n^d (i.e. equation (13)) through the generic map

$$X \mapsto \sum_i \lambda_i v(i)$$

we compute

$$\begin{aligned} R^a_{ij}&(g_1Xg_1X - Xg_1Xg_1) \\ &= q(\lambda_j - \lambda_i)(\lambda_j(q-a_{ij}) - \lambda_i(q^{-1} - a_{ij}))\begin{pmatrix} 0 & a_{ij}(-1 + a_{ij}q - q^2) \\ q & 0 \end{pmatrix}. \end{aligned}$$

The right-hand side vanishes if $a_{ij}(\lambda_i - \lambda_j) = (q^{-1}\lambda_i - q\lambda_j)$, i.e.

$$\lambda_j(q - a_{ij}) = \lambda_i(q^{-1} - a_{ij}). \tag{54}$$

Note that equation (54) agrees with equation (30) in the case $\lambda_i = q^{2x_i}$, provided that q is not a root of unity.

This means that there is a generic homomorphism from $\mathsf{G}_n^2(q, \lambda)$ to $D_n^2(q, a)$ (noting that $\lambda = \lambda(x)$ and $a = a(x)$). Great care must be taken with specific specializations, however. For example, $q = 1$ is potentially a singular case of $x_i = \frac{\ln \lambda_i}{\ln q^2}$. If we choose λ_i freely with $q = 1$ then x_i is large and every $a_{ij} = 1$ in the image. A route which sets $\lambda_i = q^{2x_i}$ (some x) *then* takes $q \to 1$ allows different a_{ij}s, but may be non-generic for λ. This means that there are important specializations in which D_n^d and its quotients (such as the blob algebra [37] in the case $d = 2$) cannot be realized as homomorphic images of specializations of G_n^d.

Of particular interest at $d = 2$ is the specialization $a_{12} = 0$. This corresponds to the specialization of the blob algebra required for the Temperley–Lieb quotient [36]. Note that the specialization $a_{12} = 0$ corresponds to $\frac{\lambda_1}{\lambda_2} = q^2$, and so is degenerate in G_n^d at $q = 1$ (i.e. G_n^d has *fewer simple modules*). That is, this specialization is one of those in which G_n^d and D_n^d behave differently. This degeneracy makes D_n^d the better candidate for physical model building.

Acknowledgments

This paper is based on some earlier unpublished notes of DL and PPM. Thanks are due from DL and PPM to DW for persuading us to complete, after an extended hiatus. Thanks are due to Rob Wilson for bringing the work of Malle and Broué to our attention, to Gunter Malle in turn for introducing us to Ariki, and finally to S Ariki for showing us his then unpublished papers with Koike. We would like to thank EPSRC for financial support of part of this project under grants GRJ25758, GRJ29923 and GRJ55069.

Appendix A. Algebraic generalities and the structure of $H_n(q)$

We use the following well known result [6, 32, 46].

Proposition 22. *Let A be a $\mathbb{C}$-algebra and A' a subalgebra. Let M be an A-module with basis B such that $M = Ab$ for each $b \in B$. Let $\cup_{i\in\Lambda} B_i = B$ be a partition of B. If each B_i is a basis for a simple A'-submodule M_i of M with $M_i \cong M_j$ only if $i = j$, then M is a simple A-module.*

Proof. Since the restriction is multiplicity free, any proper submodule M' must restrict uniquely to $\sum_{i\in\Lambda'} M_i$ as an A'-module, for some $\Lambda' \subset \Lambda$. Thus $B' = \cup_{i\in\Lambda'} B_i$ is a basis for M'. But then for $b' \in B'$ we have $M' \supseteq Ab' = M$, a contradiction. □

Proposition 23. *Let A be an algebra over a ring R, defined by generators and relations. Let B be a spanning set for A, of degree d. Suppose there exists a field $K \supset R$ such that there exists a set of inequivalent absolutely irreducible representations of A over K, whose summed squared dimensions is d. Then the K-algebra so obtained is semisimple, and A is a free R-module with basis B.*

Proof. The summed squared dimensions of inequivalent irreducibles gives a lower bound on the dimension of A over K, but since B is still spanning over K, the degree of B is an upper bound, hence the dimension is d. Thus the set of inequivalent irreducibles is complete, and there is no radical. Furthermore, B is a basis of A over K. But a linear dependence over R would imply a linear dependence over K, so we are done. □

A.1. Deformation generalities

The representation theory of $\mathbb{C}S_n$ and $\mathbb{C}C_d \wr S_n$ is well understood. We now consider the extent to which these theories inform our study of their deformations.

Let R be an algebraically closed field (it might as well be $\mathbb{C}$). Let $v = (v^1, v^2, \ldots, v^l)$ be a finite-ordered set of indeterminates and R_v be the ring of polynomials in these indeterminates with coefficients in R. Let $\mathcal{H}[v]$ be a unital finite-dimensional algebra over R_v with basis B. Let R_v^0 be the quotient field of R_v and $R(v)$ be the algebraic closure of R_v. Let $\mathcal{H}(v) = R(v) \otimes_{R_v} \mathcal{H}[v]$ be the same algebra over $R(v)$. The algebra over R obtained from $\mathcal{H}[v]$ by replacing the indeterminates v by specific elements of R, $v = v_s$ say, is here called a *specialization* of $\mathcal{H}(v)$ and is denoted $\mathcal{H}(v_s)$.

Conversely, an algebra $\mathcal{H}(v)$ is here called a *deformation* of an algebra $\mathcal{H}$ if their exists a specialization of the parameters $v \mapsto v_0$ such that $\mathcal{H}(v_0) = \mathcal{H}$. For example, $\mathcal{H}_n(q)$ over $\mathbb{C}(q)$ is a deformation of $\mathbb{C}S_n$ with $q_0 = 1$.

We want to port representation theory data between the algebras $\mathcal{H}(v)$ and $\mathcal{H}(v_s)$. This task is made difficult by the fact that there is no formal map between them (there are elements of $\mathcal{H}(v)$ for which the 'substitution' of v_s for v makes no sense). We have, rather,

$$\begin{tikzcd} & \mathcal{H}[v] \arrow[dl, "R_v \subset R(v)"'] \arrow[dr, "v \mapsto v_s"] & \\ \mathcal{H}(v) & & \mathcal{H}(v_s) \end{tikzcd} \tag{55}$$

Recall that a field F is a splitting field for a finite-dimensional algebra A if every irreducible representation of A over F is absolutely irreducible [24, section 5.3]. A splitting field F can always be found, since the algebraic closure of any initial field is splitting field, but a much smaller extension field may suffice. For example $\mathbb{Q}$ is a splitting field for S_n, and the field of fractions of $\mathbb{Z}[q]$ is one for H_n. We are interested in the situation in which the algebra may be defined over a ground ring $\mathbb{Z}[v]$ (or $\mathbb{Z}[[v]]$), say, and that on extending this ring to a splitting field F the algebra is semisimple, and hence just a sum of matrix algebras of certain dimensions; and on the other hand that on extending to $\mathbb{C}[v]$ and then specializing via $v \mapsto v_s \in \mathbb{C}$ we also have (for certain v_s) a semisimple algebra. The semisimple algebras are over different fields, but the dimension data may be compared. If the dimension data coincide we will say that algebras are *isomorphic*.

One way to make direct associations between elements of $\mathcal{H}(v)$ and $\mathcal{H}(v_s)$ in case $x \in R_v^0 \otimes_{R_v} \mathcal{H}[v] \subseteq \mathcal{H}(v)$ is to look for a pair $(a, b) \in R_v \times \mathcal{H}[v]$ such that $ax = b$ and $a|_{v_s} \neq 0$. If such a pair exists then $(a|_{v_s})^{-1} b|_{v_s} \in \mathcal{H}(v_s)$ may be naturally associated to x (this takes idempotent to idempotent for example). A more dangerous process is the attempt to extract a limit when both $a|_{v_s}$ and $b|_{v_s}$ are zero. More generally, we will say that $f \in R(v)$ is *well defined* at v_s if it takes a well defined value when regarded as a function in the usual analytical sense.

Proposition 24. *For $l = 1$, every central idempotent of an algebra $\mathcal{H}(v)$ (as above) is well defined in every semi-simple specialization of $\mathcal{H}(v)$.*

Proof: (by contradiction). Suppose central idempotent I formally blows up at some v_s where $\mathcal{H}(v_s)$ is semisimple (as if the coefficient of some basis element is $\frac{1}{v^1 - v_s^1}$ for example). Since all coefficients of $I \in \mathcal{H}(v)$ are algebraic functions each has at most an algebraic singularity in v and v_s (by the basic theorem of algebraic functions—see, e.g., Ahlfors [1] chapter VI, theorem 4). Then for some set of non-negative rational numbers p_i with $\sum_i p_i > 0$ the element

$$I' = \prod_i (v^i - v_s^i)^{p_i} I$$

(where the product is over the indeterminates in v, v^i is the ith indeterminate in v and v_s^i is its specialization) is well defined in $\mathcal{H}(v_s)$ and obeys $I'I' = 0$. For $l = 1$, p_1 can be chosen so that $I' \neq 0$. Then $\mathcal{H}(v_s)I' = I'\mathcal{H}(v_s)$ is a non-empty nilpotent double-sided ideal—a contradiction! □

Proposition 25. *For $l = 1$, if any specialization $\mathcal{H}(v_s)$ of an algebra $\mathcal{H}(v)$ is semisimple, then $\mathcal{H}(v)$ is semisimple. Further, any representation of $\mathcal{H}(v_s)$ may be obtained, up to isomorphism,* *by substituting $v = v_s$ into some representation of a generating set of elements of $\mathcal{H}(v)$ (although other specializations of $\mathcal{H}(v)$ need not be semisimple).*

Proof. Suppose $\mathcal{H}(v)$ has an element $a(v)$ in the radical. Then so is any non-vanishing scalar multiple $a'(v)$, and for some such multiple $a'(v_s)$ is non-vanishing in the radical of $\mathcal{H}(v_s)$—a contradiction. Thus $\mathcal{H}(v)$ is semisimple. Now let $1 = \sum_i I_i$ be the decomposition of the unit into primitive central idempotents of $\mathcal{H}(v)$, so the left ideal $\mathcal{H}(v)I_i$ is a direct sum of isomorphic simple left modules, each isomorphic to a full matrix algebra (see, e.g., Cohn [12] ch 5, proposition 3.6). Each I_i is well defined in every semisimple specialization of $\mathcal{H}(v)$ (see above), and each remains distinct, since $I_i I_j = \delta_{ij} I_i$, and primitive, since $R(v)$ is algebraically closed. Thus they remain a complete set.

Finally let $\{w_j : j = 1, 2, \ldots\}$ be a basis of $\mathcal{H}(v)I_i$. Here again each w_j has a scalar multiple w'_j which is finite in $\mathcal{H}(v_s)$. Now $\{w'_j\}$ is still a basis of $\mathcal{H}(v)I_i$, so its image is a spanning set in $\mathcal{H}(v_s)I_i$. Thus no ideal $\mathcal{H}I_i$ can have greater dimension in the specialization. But then to get the same total dimension (dimension of semisimple algebra over algebraically closed field = sum over i of squares of irreducible dimensions) none can have lesser dimension either. □

Corollary 25.1. *Every semisimple specialization of an algebra $\mathcal{H}(v)$ has the same structure.*

Thus to determine the structure of some $\mathcal{H}(v)$ it is sufficient to determine the structure of any semisimple specialization. For example, the structure of $H_n(q)$ over the rational field $\mathbb{C}(q)$ is isomorphic to that of the semisimple group algebra $\mathbb{C}S_n$, which is well known [13, 21] (see also section 3).

For our purposes the point is that we can use the same trick for deformations of $\mathbb{C}C_d \wr S_n$. The problem with these proofs in case $l > 1$ occurs when $I \in \mathcal{H}(v)$ diverges with two variables (say v^1, v^2) simultaneously as $v \to v_s$. In so far as this requires a coincidence, we may be guided by the likelihood that these propositions will also hold for many algebras with larger l.

Appendix B. Combinatorial approach to completeness

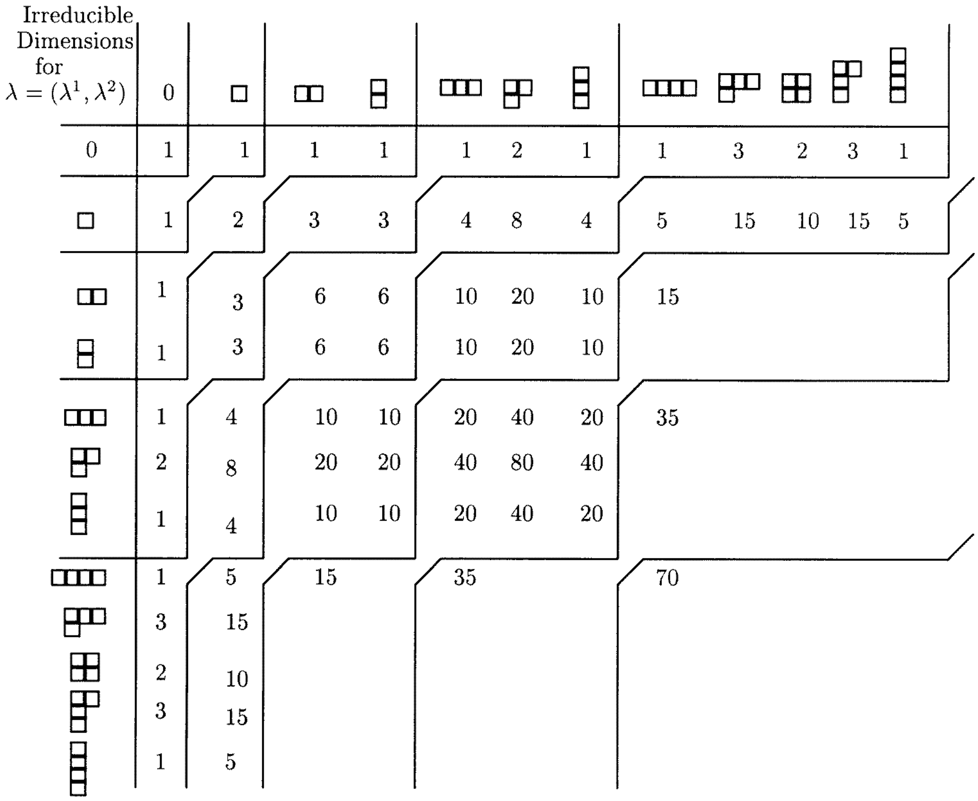
Alternatively to proposition 12, an illuminating explicit counting argument shows that the representations given are a complete set. Note

$$\dim(R^\lambda) = \prod_{i=1}^{d} \binom{n - \sum_{k=i}^{d-1} |\lambda^k|}{|\lambda^i|} \dim(\lambda^i) \tag{56}$$

and then using $\sum_{\mu \vdash n} (\dim(\mu))^2 = n!$ [44] we get, after some work,

$$\sum_\lambda (\dim(R^\lambda))^2 = n! d^n, \tag{57}$$

hence they are a complete set by equation (17). More explicitly still, for example, the table below begins to list the irreducibles by dimension for $d = 2$ and $n = 1, 2, 3, 4, \ldots$ (and selected others):



Here the dimension of the top-left-hand representation in each 'box' is $\dim(R^{(\lambda^1,\lambda^2)}) = \binom{n}{r}$ where $\lambda^1 \vdash r$. This means that the contribution to the dimension counting in equation (57) is $\binom{n}{r}^2$. But for the whole box of representations this contribution becomes $(n-r)!r!\binom{n}{r}^2$ (using equation (56) for the diagrams indexing each side of the box) so we have total algebra dimension

$$\sum_{r=0}^{n} (n-r)!r!\binom{n}{r}^2 = n!\sum_{r=0}^{n} \frac{n!}{(n-r)!r!} = n!2^n$$

(summing binomial coefficients).

For general d the key identity is

$$\sum_{\lambda \vdash n, \lambda'_1 \leqslant d} \frac{n!}{\prod_i(\lambda_i!)} = d^n$$

where the sum is over ordered d-tuples $(m_1, m_2, \ldots, m_d)$ such that $\sum_{i=1}^{d} m_i = n$ (multinomial coefficients). The 'boxes' of irreducibles are indexed by such d-tuples, and in this case the 'top-left-hand' representation in each (hypercubical) box has $\lambda = ((m_1), (m_2), \ldots, (m_d))$ with

$$\dim(R^\lambda) = \frac{n!}{\prod_{i=1}^{d} m_i!}$$

giving total dimension

$$\sum_{\lambda \vdash n, \lambda'_1 \leqslant d} \left(\prod_{i=1}^{d} m_i! \left(\frac{n!}{\prod_{i=1}^{d} m_i!}\right)^2\right) = n!d^n$$

as required.

To put this in an established context note that the blob algebra $b_n(q, q')$ [37] (of dimension $\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2$) is the quotient containing the top left representation in each box in the table above (that is, all representations of form $\lambda = ((m), (n-m))$). The obvious generalization of the blob algebra to $d > 2$ (blobs of d colours) is straightforwardly analysed using the category theory techniques of [37], however it should be noted that this is *not* a quotient of the algebras here under investigation for $d > 2$. The quadratic relation on the boundary operator is crucial for this correspondence.

References

[1] Ahlfors L V 1953 *Complex Analysis* (New York: McGraw-Hill)

[2] Alcaraz F C, Dasmahapatra S and Rittenberg V 1998 Stochastic models with boundaries and quadratic algebras *Physica* A **257** 1–9

[3] Alcaraz F C and Rittenberg V 1993 Reaction–diffusion processes as physical realizations of Hecke algebras *Phys. Lett.* B **314** 377–80

[4] Andrews G E, Baxter R J and Forrester P J *J. Stat. Phys* **35** 84

[5] Ariki S 1996 On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$ *J. Math. Kyoto University* **36** 789–808

[6] Ariki S and Koike K 1994 A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations *Adv. Math.* 216–43

[7] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[8] Bergman G M 1978 The diamond lemma for ring theory *Adv. Math.* **29** 171–218

[9] Birman J S 1975 Braids, links and mapping class groups *Annals of Mathematics Studies* vol 82 (Princeton, NJ: Princeton University Press)

[10] Broué M and Malle G 1993 Zyklotomische heckealgebren *Astérisque* **212** 119–89

[11] Cherednik I 1984 *Theor. Math. Phys.* **61** 55

[12] Cohn P M 1982 *Algebra* vol 2 (New York: Wiley) p 255

[13] Robinson G deB 1961 *Representation Theory of the Symmetric Group* (Toronto: University of Toronto Press)

[14] deVega H J and Gonzalezruiz A 1994 Boundary K-matrices for the XYZ-spin, XXZ-spin and XXX-spin chains *J. Phys. A: Math. Gen.* **27** 6129–37

[15] deVega H J and Gonzalezruiz A 1994 Exact Bethe-ansatz solution for $a(n-1)$ chains with non-$su_q(n)$ invariant open boundary-conditions *Mod. Phys. Lett.* A **9** 2207–16

[16] Dipper R, James G and Mathas A 1998 Cyclotomic q-Schur algebras *Math. Z.* **229** 385–416

[17] Dipper R, James G and Mathas A 1998 The (Q, q) Schur algebra *Proc. Lond. Math. Soc.* **77** 327–61

[18] Donkin S 1993 On tilting modules for algebraic groups *Math. Z.* **212** 39–60

[19] Ghoshal S and Zamolodchikov A 1994 Boundary S-matrix and boundary state in 2-dimensional integrable quantum-field theory *Int. J. Mod. Phys.* A **9** 4353

[20] Grimm U and Rittenberg V 1991 *Nucl. Phys.* B **354** 418

[21] Hamermesh M 1962 *Group Theory* (Oxford: Pergamon)

[22] Hoefsmit P N 1974 Representations of Hecke algebras of finite groups with BN pairs of classical type *PhD Thesis* University of British Columbia

[23] Humphreys J E 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[24] Jacobson N 1980 *Basic Algebra* vol 2 (San Francisco, CA: Freeman)

[25] Jantzen J C 1987 *Representations of Algebraic Groups* (New York: Academic)

[26] Kauffman L H 1991 *Knots and Physics* (Singapore: World Scientific)

[27] Lambropoulou S S F 1994 Solid torus links and Hecke algebras of B type *Proc. Quantum Topology* (Singapore: World Scientific) p 225

[28] Levy D 1990 Structure of Temperley–Lieb algebras and its application to 2D statistical models *Phys. Rev. Lett.* **64** 499–502

[29] Levy D and Martin P 1994 Hecke algebra solutions to the reflection equation *J. Phys. A: Math. Gen.* **27** L521–L526

[30] LimaSantos A and Ghiotto RCT 1998 A Bethe ansatz solution for the closed $U_q[sl(2)]$ Temperley–Lieb quantum spin chains *J. Phys. A: Math. Gen.* **31** 505–12

[31] Martin P P 1989 String-like lattice models and Hecke algebras *J. Phys. A: Math. Gen.* **22** 3103–12

[32] Martin P P 1991 *Potts Models and Related Problems in Statistical Mechanics* (Singapore: World Scientific)

[33] Martin P P and Rittenberg V 1992 A template for quantum spin chain spectra *Int. J. Mod. Phys.* A **7** *(suppl.1B)* 707–30

[34] Martin P P and Westbury B W 1997 On quantum spin-chain spectra and the representation theory of Hecke algebras by augmented braid diagrams *J. Phys. A: Math. Gen.* **30** 5471–95

[35] Martin P P and Woodcock D 1998 On quantum spin-chain spectra and the structure of Hecke algebras and q-groups at roots of unity *J. Phys. A: Math. Gen.* **31** 10131–54

[36] Martin P P and Woodcock D 1999 On the structure of the blob algebra *J. Algebra* at press

[37] Martin P P and Saleur H 1994 The blob algebra and the periodic Temperley–Lieb algebra *Lett. Math. Phys.* **30** 189–206

[38] Mezincescu L and Nepomechie R 1991 *J. Phys. A: Math. Gen.* **24** L17–L23

[39] Mezincescu L and Nepomechie R 1991 *Int. J. Mod. Phys.* A **6** 5231

[40] Mezincescu L and Nepomechie R 1992 *Int. J. Mod. Phys.* A **7** 5657

[41] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523

[42] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375

[43] Soergel W 1997 Charakterformeln für Kipp-Moduln über Kac–Moody-Algebren *Representation Theory* **1** 115–32

[44] Stanton D and White D 1986 *Constructive Combinatorics* (New York: UTM Schwinger)

[45] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **251** 251–80

[46] Vershik A M and Okunkov A Y 1996 An inductive method of expounding the representation theory of symmetric groups *Russ. Math. Surv.* **51** 1237–9